Toroidal and Orbifold Compactifications

环面与轨形紧化

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Abstract

摘要

In this chapter we will review toroidal and orbifold compactifications of string theory. These give rise to exact conformal field theories and, in particular, using orbifolds one can partially break supersymmetries in lower dimensions. Construction of these models and consistency conditions are discussed and some examples are provided.

本章我们将回顾弦理论的环面与轨形紧致化。这些构造能得到严格共形场论，尤其是利用轨形可以部分破缺低维时空的超对称。本文讨论了这些模型的构造与相容性条件，并给出了若干例子。

Keywords

关键词

String theory compactification

弦理论紧致化

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环面和轨形紧致化

We have seen that there are several consistent superstring theories in 10-dimensions. Type IIA and IIB give rise to theories with 32 supercharges while Heterotic and

我们已经知道，十维空间中存在多个自洽的超弦理论。IIA 型和 IIB 型超弦给出的理论具有 32 个超荷，而杂化弦和

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Type I theories give rise to theories with 16 supercharges. In order to get realistic theories, one would like to compactify these 10-dimension theories to 4-dimension. Moreover we would like to have minimal supersymmetry namely 4-supercharges in 4-dimension. In general the 6-dimension compact space needed to obtain a partially broken supersymmetric theory will be a curved space. The corresponding world sheet theory in general will be an interacting conformal field theory and typically not solvable exactly. In this chapter we will discuss examples of exact conformal theories that are solvable and one can obtain the complete spectrum of the string states and compute the correlation functions exactly.

I 型超弦给出的理论具有 16 个超荷。为了得到现实的理论，我们需要将这些十维理论紧致化到四维。此外我们希望得到最小超对称，即四维空间中保留 4 个超荷。一般来说，要得到部分破缺的超对称理论，所需的六维紧致空间是弯曲空间。对应的世界面理论通常是相互作用的共形场论，无法精确求解。本章我们将讨论可解精确共形理论的例子，这类理论可以得到弦态的完整谱，还能精确计算关联函数。

Simplest such compactifications are known as Toroidal compactification where the compact space is a flat torus. Consider a circle S^1 of radius R . This can be thought of as real line whose points are labelled by coordinate X with X being identified with $X + 2\pi nR$ for all integers n . The mode expansion for X is given by [1]

最简单的这类紧致化是环面紧致化，其紧致空间是平环面。考虑一个半径为 R 的圆 S^1 。它可以看作实直线，其上的点由坐标 X 标记，对任意整数 n ， X 与 $X + 2\pi nR$ 等价。X 的模展开式为 [1]

$$X(\sigma, t) = x_0 + \frac{p}{2R} t + wR\sigma + \sum_{n \neq 0} \left(\alpha_n e^{in(t+\sigma)} + \widetilde{\alpha}_n e^{in(t-\sigma)} \right) \tag{1}$$

where w is the winding number (i.e., it counts the number of times the closed string wraps around the circle S^1) and momenta are labelled by integers p (this is analogous to a point particle case where the momenta of the particle moving on a circle is given by an integer times $1/R$). The 1-loop partition function for this system is

其中 w 是绕数 (即它记录闭弦绕圆 S^1 缠绕的次数)，动量由整数 p 标记 (这类似于点粒子的情形: 在圆上运动的粒子的动量等于整数乘以 $1/R$)。该系统的单圈配分函数为

$$Z = \frac{1}{\eta \bar{\eta}} \sum q^{\frac{1}{2}(wR + p/(2R))^2} \bar{q}^{\frac{1}{2}(wR - p/(2R))^2} \tag{2}$$

where the sum is over all integers p and w . Note that this is a 2-dimension lattice labelled by integers p and w . We can rewrite this equation is a more suggestive way.

其中求和遍历所有整数 p 和 w 。注意这是一个由整数 p 和 w 标记的二维格点。我们可以将该方程改写为更直观的形式。

$$Z = \frac{1}{\eta \bar{\eta}} \sum_{(P_L, P_R) \in \Gamma_{1,1}} q^{\frac{1}{2} P_L^2} \bar{q}^{\frac{1}{2} P_R^2} \tag{3}$$

where $P_L = wR + p/(2R)$ and $P_R = wR - p/(2R)$. It is clear that $P_L^2 - P_R^2 = 2wp$ is an even number. Thus $\Gamma_{1,1}$ is a 2-dimensional lattice which has the property that length square of any vector in $\Gamma_{1,1}$ is even with signature $(+, -)$. Consider now the dual lattice of $\Gamma_{1,1}$ (where the inner product is again with the same signature). If (y_L, y_R) is in the dual lattice, then it must have integral dot product with all the vectors of $\Gamma_{1,1}$. This implies that $R(y_L - y_R)$ and $(y_L + y_R)/(2R)$ must be integers say n_1 and n_2 . This in turn implies that $y_L = n_1/(2R) + n_2 R$ and $y_R = n_2 R - \ n_1/(2R)$ and therefore $(y_L, y_R) \in \Gamma_{1,1}$. This also implies that the partition function Z is modular invariant under $SL(2, Z)$ transformation $\tau \to (a\tau + b)/(c\tau + d)$ where a, b, c, d are integers and satisfy $ad - bc = 1$. This group is generated by two elements T -transformation $\tau \to \tau + 1$ and S -transformation $\tau \to -1/\tau$. Under T -transformation, the lattice terms of Z pick phase $e^{\pi i (P_L^2 - P_R^2)}$ and therefore T -invariance implies $\Gamma_{1,1}$ must be even (with Lorentzian signature). Similarly, using Poisson resummation one can show that S invariance implies that $\Gamma_{1,1}$ is self dual (this is shown in Appendix A in a more general context).

其中 $P_L = wR + p/(2R)$ 和 $P_R = wR - p/(2R)$ 。显然 $P_L^2 - P_R^2 = 2wp$ 为偶数，因此 $\Gamma_{1,1}$ 是一个二维格，满足格中任意向量的长度平方为偶数，符号差为 $(+,-)$ 。现在考虑 $\Gamma_{1,1}$ 的对偶格 (其内积与原格符号差相同)。若 (y_L, y_R) 属于对偶格，则它与 $\Gamma_{1,1}$ 中所有向量的点积必为整数。由此可得 $R(y_L - y_R)$ 和 $(y_L + y_R)/(2R)$ 必为整数，记为 n_1 和 n_2 ，进而可得 $y_L = n_1/(2R) + n_2 R$ 、$y_R = n_2 R - n_1/(2R)$ ，因此 $(y_L, y_R) \in \Gamma_{1,1}$ 。这也说明配分函数 Z 在 $SL(2,Z)$ 变换 $\tau \to (a\tau + b)/(c\tau + d)$ 下具有模不变性，其中 a,b,c,d 为整数且满足 $ad - bc = 1$ 。该模群由两个元素生成: T 变换 $\tau \to \tau + 1$ 和 S 变换 $\tau \to -1/\tau$ 。在 T 变换下，Z 的格项会引入相位 $e^{\pi i (P_L^2 - P_R^2)}$ ，因此 T 不变性要求 $\Gamma_{1,1}$ 必为偶格 (带洛伦兹符号差)。类似地，利用泊松求和公式可证明 S 不变性要求 $\Gamma_{1,1}$ 是自对偶格 (该结论在附录 A 的更一般框架下给出了证明)。

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[1] Throughout this chapter we will use the units where $\alpha' = 1/2$.

[1] 本章我们统一采用单位制，其中 $\alpha' = 1/2$ 。

We can generalize this discussion (Refs. [1] and [2]) and consider $(d_L + d_R)$ dimensional lattice Γ_{d_L,d_R} with d_L positive and d_R negative signatures (i.e., d_L left moving and d_R right moving directions). If we are considering Type II theories, then $d_L = d_R$ while for Heterotic theory $d_L = d_R + 16$. The one-loop partition function of this CFT is

我们可以将上述讨论推广 (参考文献 [1] 和 [2])，考虑 $(d_L + d_R)$ 维格 Γ_{d_L,d_R} ，其正符号差方向数为 d_L ，负符号差方向数为 d_R (即 d_L 个左动方向，d_R 个右动方向)。对于 II 型理论，有 $d_L = d_R$ ，而对于杂弦理论，有 $d_L = d_R + 16$ 。该共形场论的单圈配分函数为

$$Z = \frac{1}{\eta^{d_L} \bar{\eta}^{d_R}} \sum_{(P_L, P_R) \in \Gamma_{d_L,d_R}} q^{\frac{1}{2} P_L^2} \bar{q}^{\frac{1}{2} P_R^2} \tag{4}$$

Under T -transformation the lattice terms of Z pick up phase $e^{\pi i (P_L^2 - P_R^2)}$ and therefore T -invariance implies that $P_L^2 - P_R^2$ must be even, i.e., Γ_{d_L,d_R} must be even. It is shown in the Appendix A, by doing a Poisson resummation of the lattice sum, that S -invariance implies that Γ must be also self dual.

在 T 变换下，Z 的格项会引入相位 $e^{\pi i (P_L^2 - P_R^2)}$ ，因此 T 不变性要求 $P_L^2 - P_R^2$ 必为偶格，即 Γ_{d_L,d_R} 必为偶数。附录 A 中通过对格求和做泊松重求和证明，S 不变性要求 Γ 同时为自对偶格。

Thus modular invariance implies that Γ_{d_L,d_R} must be even and self dual. Fortunately, there is a general theorem which says that such even and self dual lattices exist if and only if $d_L - d_R = 8n$ for some integer n . In the string theory context, as mentioned above, $d_L - d_R = 0$ for Type IIA and IIB, while for Heterotic string $d_L - d_R = 16$. So in both Type IIA, B and Heterotic string such even self dual lattices exist. Moreover, if d_L and d_R both are non-zero (which is the case that appears in the context of Toroidal compactification), then such even self dual lattice is unique up to $SO(d_L, d_R)$ transformations. In string theory context, the spectrum depends on P_L^2 . This is because the masses of the string states are given by

因此模不变性表明 Γ_{d_L,d_R} 必须是偶自对偶格。幸运的是, 存在一条一般性定理: 这类偶自对偶格存在当且仅当对某个整数 n 有 $d_L - d_R = 8n$ 。在弦论框架下, 如上所述, IIA 与 IIB 型弦满足 $d_L - d_R = 0$,而杂化弦满足 $d_L - d_R = 16$ 。因此无论是 IIA、IIB 型弦还是杂化弦，都存在这类偶自对偶格。此外,如果 d_L 和 d_R 均非零 (这正是环面紧化情形下会出现的情况)，那么这类偶自对偶格在 $SO(d_L, d_R)$ 变换下是唯一的。在弦论框架下，能谱依赖于 P_L^2 。这是因为弦态的质量由下式给出

$$m^2 = N + \frac{1}{2}P_L^2 - a_L = \widetilde{N} + \frac{1}{2}P_R^2 - a_R \tag{5}$$

where N and $\widetilde{N}$ are the left and right moving oscillator numbers (that include bosonic modes as well as fermionic modes in case of fermionic string) and a_L and a_R are the zero point shifts. What this means is that even though the difference $P_L^2 - P_R^2$ is invariant under $SO(d_L, d_R)$ transformations, the spectrum will be invariant only under $SO(d_L) \times SO(d_R)$ subgroup of $SO(d_L, d_R)$. Thus the moduli space of such compactifications is

其中 N 和 $\widetilde{N}$ 分别是左行和右行振荡子数，对于费米弦而言，还同时包含玻色模与费米模，a_L 和 a_R 是零点移位。这意味着，即使差 $P_L^2 - P_R^2$ 在 $SO(d_L, d_R)$ 变换下不变，能谱也仅在 $SO(d_L, d_R)$ 的 $SO(d_L) \times SO(d_R)$ 子群下不变。因此这类紧化的模空间是

$$M_{d_L,d_R} = \frac{SO(d_L, d_R)}{SO(d_L) \times SO(d_R)} \tag{6}$$

up to some discrete identifications. The dimension of $M_{d_L,d_R} = (d_L + d_R)(d_L + \ d_R - 1)/2 - d_L(d_L - 1)/2 - d_R(d_R - 1)/2 = d_L d_R$. Thus the moduli space of this toroidal compactification involves $d_L d_R$ number of parameters. We will return to the physical interpretation of these parameters later.

相差一些离散等同。模空间的维数为 $M_{d_L,d_R} = (d_L + d_R)(d_L + \ d_R - 1)/2 - d_L(d_L - 1)/2 - d_R(d_R - 1)/2 = d_L d_R$ 。因此这种环面紧化的模空间包含 $d_L d_R$ 个参数。我们后续会再讨论这些参数的物理解释。

First let us consider some examples. In Type II theories, for $d_L = d_R = n$, if we take the lattice consisting of the conjugacy classes of D_n weight lattices, i.e., the root lattice of D_n denoted by Sc_n , the two spinor classes Sp_n and Sp'_n and the vector class V_n , then we can take $\Gamma_{n,n} = (Sc_n, Sc_n) + (V_n, V_n) + (Sp_n, Sp_n) + (Sp'_n, Sp'_n)$. Recall that length squares of weights in Sc_n are even numbers, V_n are 1 mod even numbers and Sp_n and Sp'_n are $n/4$ mod even numbers and the dot products of weights in Sc_n with all the four classes is integer (this is just the statement that the weight lattice of D_n is dual to Sc_n), the dot product of weights in V_n class with that of V_n , Sp_n , and Sp'_n classes are, respectively, $1, 1/2$, and $1/2$ mod integers, and finally the dot product of weights of Sp_n with Sp'_n class is $(n-2)/4$ mod integers. From this we see that $\Gamma_{n,n}$ defined above is even (recalling that the signature is n pluses and n minuses) and moreover the dot products of any two vectors in $\Gamma_{n,n}$ are integers. Finally to show that it is self dual, the dual lattice must necessarily be a subset of $(Sc_n + V_n + Sp_n + Sp'_n, Sc_n + V_n + Sp_n + Sp'_n)$ since the dot product with arbitrary vectors in (Sc_n, Sc_n) must be integer. By using the dot products of different classes of D_n given above, one can check that only vectors that are in correlated classes give integer dot products with $\Gamma_{n,n}$. All the states carrying non-zero lattice momenta will be massive in Type II.

首先我们来看几个例子。对于 II 型理论，在 $d_L = d_R = n$ 中，如果我们取由 D_n 权格的共轭类构成的格，也就是记为 Sc_n 的 D_n 根格，加上两个旋量类 Sp_n 和 Sp'_n 以及矢量类 V_n，我们就可以得到 $\Gamma_{n,n} = (Sc_n, Sc_n) + (V_n, V_n) + (Sp_n, Sp_n) + (Sp'_n, Sp'_n)$。我们知道，$Sc_n$ 中权的长度平方是偶数，V_n 中权的长度平方模偶数余 1，Sp_n 和 Sp'_n 中权的长度平方模偶数余 $n/4$；且 Sc_n 中权与所有四个类中权的点积都是整数 (这正说明 D_n 的权格是 Sc_n 的对偶格)；V_n 类中权与 V_n、Sp_n、Sp'_n 类中权的点积分别模整数余 $1, 1/2$ 和 $1/2$；最后，Sp_n 类与 Sp'_n 类中权的点积模整数余 $(n-2)/4$。由此我们可以看出，上文定义的 $\Gamma_{n,n}$ 是偶格 (注意符号是 n 个正号加 n 个负号)，并且 $\Gamma_{n,n}$ 中任意两个向量的点积都是整数。下面证明它是自对偶格: 由于和 (Sc_n, Sc_n) 中任意向量的点积必须为整数，因此对偶格必然是 $(Sc_n + V_n + Sp_n + Sp'_n, Sc_n + V_n + Sp_n + Sp'_n)$ 的子集。利用上文给出的 D_n 不同类之间的点积可以验证，只有同类关联中的向量才能和 $\Gamma_{n,n}$ 得到整数点积。所有携带非零格动量的态在 II 型理论中都是有质量的。

The more interesting case is that of toroidal compactification in Heterotic string. Consider $d_R = n$ and $d_L = n + 16$. This gives $10 - n$ non-compact directions. An example of even self dual lattice can be obtained by correlating classes of D_{n+16} with those of D_n, i.e., the lattice is $\Gamma_{n+16,n} = (Sc_{n+16}, Sc_n) + (V_{n+16}, V_n) + (Sp_{n+16}, Sp_n) + \left(Sp'_{n+16}, Sp'_n\right)$. One can easily verify that this gives an even self dual lattice by repeating the arguments of the last paragraph. Now let us look at the massless states. In the right moving sector, $P_R \neq 0$ states are massive as minimum value of $N_R - a_R$ in both NS and R sectors is zero (after doing the GSO projection as seen already in the 10-dimension theory). Thus massless states will necessarily have $P_R = 0$. In the left moving sector $a_L = -1$, so the massless states can be (1) $N_L = 1$ and $P_L = 0$ or (2) $N_L = 0$ and $P_L \neq 0$ such that $P_L^2 = 2$. States with $N_L = 1$ and $P_L = 0$ just give rise to the dimensional reduction of the 10-dimension metric, antisymmetric tensor, dilaton, and the Cartan directions of the original $SO(32)$ or $E_8 \times E_8$ (i.e., $16U(1)$ gauge fields) and their fermionic partners. The states with $N_L = 0$ and $P_L^2 = 2$ give rise to massless states that are charged with respect to the Cartan directions. In the example given above, these are just roots of D_{n+16}. The resulting gauge group in $(10-n)$ dimensional theory is $SO(32+2n)$. By continuous boosting, generally such states will develop non-zero P_R and therefore these states become massive. So at a generic point in moduli space the gauge group will be $U(1)^{16+n}$ but there will be subspaces of lower dimension in the moduli space where there is enhanced gauge symmetry (always keeping the total rank $(16+n)$ coming from the left movers).

更有趣的情况是杂化弦的环面紧致化。考虑 $d_R = n$ 和 $d_L = n + 16$，由此得到 $10 - n$ 个非紧致方向。我们可以通过将 D_{n+16} 的类与 D_n 的类关联得到一个偶自对偶晶格的例子，即该晶格为 $\Gamma_{n+16,n} = (Sc_{n+16}, Sc_n) + (V_{n+16}, V_n) + (Sp_{n+16}, Sp_n) + \left(Sp'_{n+16}, Sp'_n\right)$。重复上一段的论证即可轻松验证这确实是一个偶自对偶晶格。现在我们来看无质量态: 在右行 sector 中，$P_R \neq 0$ 态是有质量的，因为 NS 区和 R 区中 $N_R - a_R$ 的最小值都是零 (完成 10 维理论中已经介绍过的 GSO 投影后)。因此无质量态必然满足 $P_R = 0$。在左行 sector 中 $a_L = -1$，因此无质量态可以分为两类:(1) $N_L = 1$ 和 $P_L = 0$，或 (2) $N_L = 0$ 和 $P_L \neq 0$ 满足 $P_L^2 = 2$。带有 $N_L = 1$ 和 $P_L = 0$ 的态恰好给出 10 维度规、反对称张量、dilation，以及原 $SO(32)$ 或 $E_8 \times E_8$ 的卡坦方向 (即 $16U(1)$ 规范场) 的维约化结果，还有它们的费米子伙伴。带有 $N_L = 0$ 和 $P_L^2 = 2$ 的态会给出对卡坦方向带荷的无质量态，在上述例子中，这些正好是 D_{n+16} 的根。最终得到的 $(10-n)$ 维理论的规范群为 $SO(32+2n)$。一般来说，通过连续 boost，这类态会获得非零的 P_R，因此变成有质量态。所以在模空间的一般点上，规范群为 $U(1)^{16+n}$，但模空间中存在低维子空间，其中会出现增强规范对称性 (总秩始终保持为左行态给出的 $(16+n)$)。

To see this more explicitly, let us denote by μ, υ the directions in the $10 - n$ non-compact space, i, j

, etc. the n dimensional compact direction and I, J as the 16 Cartan directions of the original $SO(32)$ or $E_8 \times E_8$. Then the massless bosonic states (i.e., in the right moving NS sector) with $N_L = 1$ and $P_L = 0$ are $\alpha_{-1}^{\mu} \tilde{b}_{1/2}^{v} |0>$ (which give the metric, antisymmetric tensor, and the dilaton in $(10-n)$ non-compact space-time), $\alpha_{-1}^{\mu} \tilde{b}_{1/2}^{i} \big|0>$ (which gives graviphotons) , $\alpha_{-1}^{j} \tilde{b}_{1/2}^{v} \big| 0>$ (which give $nU(1)$ gauge fields, $\alpha_{-1}^{j} \tilde{b}_{1/2}^{i} |0>$ (which gives $n(n+$ that go into the definition of n -dimensional lattice defining T^n and the $n(n-1)/2$ scalars coming from the antisymmetric tensor B_{ij}), $\alpha_{-1}^{I} \tilde{b}_{1/2}^{v} \mid 0>$ (the sixteen $U(1)$ gauge fields) and $\alpha_{-1}^{I} \tilde{b}_{1/2}^{i} \mid 0>$ (the $16n$ scalars). Thus apart from the graviphotons we have $16 + nU(1)$ gauge fields. The massless states coming from $N_L = 0$ and lattice vectors $P = (P_L, P_R)$ with $P_R = 0$ and $P_L^2 = 2$ give rise to massless states $\tilde{b}_{-1/2}^{v} \big|P>$ (gauge fields) and $\tilde{b}_{-1/2}^{i} \big| P>$ (the corresponding scalar partners). Note that these gauge fields and scalars (denoting them by A_{μ}^{P} and A_{i}^{P}) are charged under the $(16+n)\,U(1)$ gauge fields. This follows from the singular terms in the OPE among the left moving fields (here A refers to the indices i and I , i.e., A runs over $(16+n)$ indices)

为了更清楚地说明这一点，我们将 μ, v 记作 $10-n$ 维非紧致空间的方向，i, j 等记作 n 维紧致方向，I, J 记作原 $SO(32)$ 或 $E_8 \times E_8$ 的 16 个嘉当方向。此时，满足 $N_L = 1$ 和 $P_L = 0$ 的无质量玻色态 (即右运动 NS 扇区中的态) 为: $\alpha_{-1}^{\mu} \tilde{b}_{1/2}^{v} |0>$ (它们给出 $(10-n)$ 维非紧致时空的度规、反对称张量和胀子)，给出 $nU(1)$ 规范场的 $\alpha_{-1}^{\mu} \tilde{b}_{1/2}^{i} \big|0>$ (which gives graviphotons) , $\alpha_{-1}^{j} \tilde{b}_{1/2}^{v} \big| 0>$ (，参与定义刻画 T^n 的 n 维格的 $\alpha_{-1}^{j} \tilde{b}_{1/2}^{i} |0>$ (which gives $n(n+1)/2$ metric components)|，以及来自反对称张量 B_{ij}), $\alpha_{-1}^{I} \tilde{b}_{1/2}^{v} \mid 0>$ (十六个 $U(1)$ 规范场) 的 $n(n-1)/2$ 标量和 $\alpha_{-1}^{I} \tilde{b}_{1/2}^{i} \mid 0>$ ($16n$ 个标量)。因此除引力光子外，我们得到 $16 + nU(1)$ 个规范场。来自 $N_L = 0$ 和满足 $P_R = 0$ 、$P_L^2 = 2$ 的格矢量 $P = (P_L, P_R)$ 的无质量态给出无质量态 $\tilde{b}_{-1/2}^{v} \big|P>$ (gauge fields) and $\tilde{b}_{-1/2}^{i} \big| P>$ (对应标量伴子)。注意这些规范场和标量 (我们记为 A_{μ}^{P} 和 A_{i}^{P}) 在 $(16+n)\,U(1)$ 规范场下带荷。这可由左运动场之间算子乘积展开中的奇异项推出 (此处 A 指指标 i 和 I ，即 A 遍历 $(16+n)$ 个指标)

$$\partial X^A(z)\, e^{iP.X}(w) = i \frac{P^A}{z-w} e^{iP.X}(w) + \ldots \tag{7}$$

where dots represent nonsingular terms. Integrating z around closed contour surrounding w gives the charge P^A . If we have two such massless states P and P' the leading term in the OPE is

其中点代表非奇异项。对包围 w 的闭合围道积分 z 可得荷 P^A 。若存在两个这样的无质量态 P 和 P' ，算子乘积展开的领头项为

$$e^{iP.X}(z)\, e^{iP'.X}(w) = (z-w)^{P.P'} e^{i(P+P').X}(w) + \ldots \tag{8}$$

Now $P.P'$ must be an integer since the lattice $\Gamma_{16+n,n}$ is self dual. Furthermore since $P_R = P_R' = 0$ and $P_L^2 = P'^2_L = 2, P.P'$ must be ± 2 (when $P' = \pm P$) or ± 1 or zero. When $P.P' = -1$ there is a singular term in the OPE and integrating z along a contour encircling w gives $e^{i(P+P').X}$ which is again massless since $(P+P')^2 = P^2 + P'^2 + 2P \cdot P' = 2 + 2 + 2(-1) = 2$. Length squares of such vectors are always 2, therefore the algebra we get is a simply laced Lie algebra, i.e., it must consist of A, D , and E series in the Dynkin classification with total rank $16+n$.

由于格 $\Gamma_{16+n,n}$ 是自对偶的，因此 $P.P'$ 必须为整数。此外，由于 $P_R = P'_R = 0$ 和 $P_L^2 = P'^2_L = 2, P.P'$ 必须满足 ± 2 (当 $P' = \pm P$ 时)，因此只能取 ± 1 或零。当 $P.P' = -1$ 时，OPE 中会出现一个奇异项，将 z 沿环绕 w 的围道积分得到 $e^{i(P+P').X}$ ，由于 $(P+P')^2 = \ P^2 + P'^2 + 2P \cdot P' = 2+2+2(-1) = 2$ ，该粒子仍为无质量粒子。这类向量的长度平方恒为 2，因此我们得到的代数是单根系李代数，即在邓金分类中，它由 A, D 和 E 系列构成，总秩为 $16+n$ 。

Thus the massless lattice state with $P_R = 0$ and $P_L^2 = 2$ gives rise to non-abelian gauge group with rank $(16+n)$. In fact this a special case of an algebra called Kac-Moody algebra

因此，满足 $P_R = 0$ 和 $P_L^2 = 2$ 的无质量格态会生成秩为 $(16+n)$ 的非阿贝尔规范群。实际上这是卡茨-穆迪代数的一个特例

$$J_a(z) J_b(w) = \frac{k g_{ab}}{(z-w)^2} + \frac{f^c_{ab} J_c(w)}{z-w} + \cdots \tag{9}$$

where f^c_{ab} are the structure constants of a Lie Algebra and g_{ab} is the metric where k is the central charge of the Kac-Moody algebra. The algebra described in Eqs. (7) and (8) define a Kac-Moody algebra with $k = 1$ and for simply laced Lie algebras.

其中 f^c_{ab} 是李代数的结构常数，g_{ab} 是度量，k 是卡茨-穆迪代数的中心荷。式 (7) 和式 (8) 描述的代数定义了满足 $k = 1$ 的卡茨-穆迪代数，且适用于单根系李代数。

Returning to physical interpretation of the continuous parameters $\left(n^2\right.$ in Type II theories and $(16+n)n$ in Heterotic string theory), they must correspond to giving expectation values to massless scalar states. There are $n(n+1)/2$ massless scalars from the metric G_{ij} and $n(n-1)/2$ massless scalars from the antisymmetric tensor B_{ij} . In Type II theories these are the only massless scalars coming from NS-NS sector. Constant vacuum expectation values of these fields are clearly flat directions of the potential. These vevs give a total of n^2 parameters.

回到连续参数的物理解释:II 型理论中的 $\left(n^2\right.$ ，以及杂弦理论中的 $(16+n)n$)，它们必然对应无质量标量态的真空期望值。其中度规场 G_{ij} 贡献 $n(n+1)/2$ 个无质量标量，反对称张量场 B_{ij} 贡献 $n(n-1)/2$ 个无质量标量。在 II 型理论中，这些是 NS-NS 扇区仅有的无质量标量。这些场的常数真空期望值显然是势能的平坦方向，这些真空期望值总共给出 n^2 个参数。

In the Heterotic theory, apart from the massless scalars G_{ij} and B_{ij} , we have also the gauge fields A^a_i where a runs over $SO(32)$ or $E_8 \times E_8$ gauge group. However the $F^a_{i,j}$ includes a commutator term $f^a_{bc} A^b_i A^c_j$ so the flat directions of the potential will involve only constant vevs of gauge fields (up to gauge transformations) along the Cartan directions. This gives $16n$ parameters. So in heterotic string the deformations parameters are $n^2 + 16n = n(16+n)$. In Ref. [2], it is shown that the one-loop path integral on world sheet torus, in the presence of constant G_{ij}, B_{ij} , and A_i along Cartan directions of $E_8 \times E_8$ or $SO(32)$, indeed gives rise to even self dual lattices $\Gamma_{16+n,n}$ involving $(16+n)n$ independent parameters.

在杂化弦理论中，除无质量标量场 G_{ij} 和 B_{ij} 外，我们还得到规范场 A_i^a，其中 a 遍及 $SO(32)$ 或 $E_8 \times E_8$ 规范群。但 $F_{i,j}^a$ 包含对易子项 $f_{bc}^a A_i^b A_j^c$，因此势的平坦方向仅涉及嘉根子方向上规范场的常数真空期望值(在规范变换下意义上)，由此得到 $16n$ 个参数。因此杂化弦的形变参数共有 $n^2+16n = n(16+n)$ 个。参考文献 [2] 已证明，当世界面环面上的单圈路径积分存在常数 G_{ij}, B_{ij} 与沿 $E_8 \times E_8$ 或 $SO(32)$ 嘉根子方向的 A_i 时，确实会生成包含 $(16+n)n$ 个独立参数的偶自对偶格 $\Gamma_{16+n,n}$。

Orbifold Compactifications

轨形紧致化

Toroidal compactification, discussed above, does not break supersymmetry even partially. For example, a $T^6 = R^6/\Gamma_6$ compactification in type II theories will give rise to $N = 8$ supersymmetric theory and in heterotic theory it gives rise to $N = 4$ theory in 4-dimension. In order to get a semi-realistic theory in 4-dimension, one would like to get $N = 1$ supersymmetry in 4-dimension. In order to achieve this, some of the Ramond spinors coming from the right moving sector must be projected out. This can be achieved by modding out the toroidal theory by a discrete rotation group acting in the compact space (Refs. [3] and [4]). To see this in some detail, the fermions in the light cone gauge transform as $SO(8)$ spinor or spinor' representation depending on the GSO projection. Expressing the weights in terms of $SO(2)^4$ subalgebra of $SO(8)$ they are $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$ with the total number of minus signs restricted to even or odd depending on whether the fermion is spinor or spinor'. We will from now on choose the fermion to be spinor, that is, there are even number of minus signs. We can take say the first $SO(2)$ to be rotation in the transverse direction of the non-compact 4-dimension space, and the last $3SO(2)$ as rotation in 3 complex planes in the 6-dimension compact space (for us it will be flat toroidal space). Denoting $T^6 = R^6/\Gamma_6$ where Γ_6 is a 6-dimension lattice the rotation must be an automorphism of Γ_6 . Let us denote by $g = (\theta, \upsilon)$ the automorphism where θ is a rotation and υ is a shift and let n be the order of g , i.e., $g^N = 1$. Let us denote the eigenvalues of θ along three planes as $\left(e^{2\pi i a_1/N}, e^{2\pi i a_2/N}, e^{2\pi i a_3/N}\right)$, where a_1, a_2 , and a_3 are integers, then g acting on spinor will pick up a phase $e^{2\pi i \frac{1}{2}\frac{(\pm a_1 \pm a_2 \pm a_3)}{N}}$. Thus for some supersymmetry to survive we have the condition $a_1 \pm a_2 \pm a_3$ must be a multiple of $2N$ for some choice of signs. Without losing generality we can set the condition $a_3 = -(a_1 + a_2)$. This will ensure that $\pm\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is invariant under g . The shift υ can be in the left moving Γ_{16} lattice in Heterotic context subject to the condition that $n\upsilon \in \Gamma_{16}$. In the Type II context if $a_2 = -a_1$ so that $a_3 = 0$ then the shift υ can also be in the third plane satisfying the condition $n\upsilon \in \Gamma_6$. Here we will focus on Heterotic theory.

前文讨论的环面对称紧致化不会破坏超对称，哪怕部分破坏都不会发生。例如，II 型理论中的 $T^6 = R^6/\Gamma_6$ 紧致化会得到 $N = 8$ 维超对称理论，杂弦理论中则会得到四维下的 $N = 4$ 理论。为得到半现实的四维理论，我们希望四维下拥有 $N = 1$ 超对称。要实现这一点，必须投影掉右动区产生的部分拉蒙德旋量。这可以通过对环面理论模去紧致空间上的离散旋转群实现 (参考文献 [3] 和 [4])。我们来详细说明: 光锥规范下的费米子根据 GSO 投影变换为 $SO(8)$ 旋量或旋量' 表示。将权用 $SO(8)$ 的 $SO(2)^4$ 子代数表示后，它们是 $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$ ，负号的总个数限制为偶数或奇数，具体取决于费米子是旋量还是旋量'。下文我们选定费米子为旋量，即负号个数为偶数。我们可以设前 $SO(2)$ 个是四维非紧致空间横方向的旋转，后 $3SO(2)$ 个是六维紧致空间 (对我们而言是平坦环面空间) 三个复平面上的旋转。记 $T^6 = R^6/\Gamma_6$ ，其中 Γ_6 是六维格，旋转必须是 Γ_6 的自同构。我们用 $g = (\theta, \upsilon)$ 表示该自同构，其中 θ 是旋转，υ 是平移，令 n 为 g 的阶，即 $g^N = 1$ 。我们将 θ 沿三个平面的本征值记为 $\left(e^{2\pi i a_1/N}, e^{2\pi i a_2/N}, e^{2\pi i a_3/N}\right)$ ，其中 a_1, a_2 、a_3 是整数，那么作用在旋量上的 g 会给出相位 $e^{2\pi i \frac{1}{2}\frac{(\pm a_1 \pm a_2 \pm a_3)}{N}}$ 。因此，要让部分超对称保留下来，我们需要满足条件: 对某些符号选择，$a_1 \pm a_2 \pm a_3$ 必须是 $2N$ 的倍数。在不失一般性的前提下我们可以设条件为 $a_3 = -(a_1 + a_2)$ 。这可以保证 $\pm\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ 在 g 作用下不变。在杂弦理论框架下，平移 υ 可以位于左动 Γ_{16} 格上，满足条件 $n\upsilon \in \Gamma_{16}$ 。在 II 型理论框架下，如果 $a_2 = -a_1$ 即 $a_3 = 0$ ，那么平移 υ 也可以位于第三个复平面上，满足条件 $n\upsilon \in \Gamma_6$ 。本文我们聚焦于杂弦理论。

To obtain the Hilbert space of states in the orbifold theory, we start with the Hilbert space H of states that were there originally in the Heterotic string before modding out by the orbifold group G . In the orbifold theory only the states in H that are g invariant will survive, denoted by H_0 . Clearly $H_0 = \frac{1}{N}\left(1 + g + g^2 + \ldots + \; g^{N-1}\right) H$. But this is not the whole story. Since $X \in T^6$ is identified with $\theta X \in T^6$ we can have closed string configurations which satisfy for $k = 0, \ldots, N-1$

为了得到轨形理论中的态希尔伯特空间，我们从杂化弦模去轨形群 G 前原本存在的态希尔伯特空间 H 出发。在轨形理论中，只有 H 里满足 g 不变性的态能够保留，记作 H_0 。显然有 $H_0 = \frac{1}{N}\left(1 + g + g^2 + \ldots + \; g^{N-1}\right) H$ 。但这并非全部情况。由于 $X \in T^6$ 与 $\theta X \in T^6$ 等价，我们可以得到满足 $k = 0, \ldots, N-1$ 条件的闭弦构型

$$X(\sigma + 2\pi, t) = \theta^k X(\sigma, t) \mod \Gamma_6$$

$$\psi(\sigma + 2\pi, t) = \mp\theta^k \psi(\sigma, t) \tag{10}$$

where $\mp$ is for NS and R sectors. For $k = 0$, the condition above was just the one for toroidal theory, i.e., the theory before orbifolding and the corresponding Hilbert space is what we denoted by H , but $k \neq 0$ are new states that were not present in H . We will denote by H_{g^k} as corresponding Hilbert spaces. Furthermore, as in the case of H_0 , we need to project the states in H_{g^k} to g -invariant states. Such g -invariant states form subspace H_k of H_{g^k} and are $H_k = \frac{1}{n}\left(1 + g + g^2 + \ldots + g^{N-1}\right) H_{g^k}$. Note that H_{g^k} comes with certain multiplicity labelled by possible positions of the center of mass of X which are fixed points under the action of θ^k .

其中 $\mp$ 对应 NS sector 和 R sector。对 $k=0$ 而言，上述条件就是环面对应理论，也就是轨形化前理论的条件，对应的希尔伯特空间就是我们记作 H 的结果，但 $k\neq 0$ 是不存在于 H 中的新态。我们将对应的希尔伯特空间记作 H_{g^k} 。此外，和 H_0 的情况一样，我们需要将 H_{g^k} 中的态投影到 g 不变态上。这类 g 不变态构成 H_{g^k} 的子空间 H_k ，且为 $H_k=\frac{1}{n}\left(1+g+g^2+\ldots+g^{N-1}\right)H_{g^k}$ 。注意 H_{g^k} 带有由质心可能位置标记的多重度，这些位置是 θ^k 作用下的不动点 X

Let us consider some examples. Consider $T^2=R^2/\Gamma_2$ where Γ_2 is a 2- dimension lattice generated by two vectors v_1 and v_2 and take a Z_2 action for θ namely $\theta X=-X$ where X is vector in $R^2 . \theta X=X \bmod \Gamma$ are called the fixed points of θ . One can easily see that there are four fixed points in this example, $X=0, v_1/2, v_2/2,\left(v_1+v_2\right)/2$. Consider another example where v_1 and v_2 have same length and are perpendicular to each other (i.e., the lattice is a square lattice), then a Z_4 rotation by $\pi/2$ will be an automorphism of this lattice. Explicitly θ takes $\left(v_1, v_2\right) \rightarrow\left(v_2,-v_1\right)$. The equation $\theta X=X \bmod \Gamma$ has now only two solutions (modulo lattice vectors) and these are $X=0,\left(v_1+v_2\right)/2 \bmod \Gamma$. Yet another example is when v_1 and v_2 are of equal length and the angle between them is $\pi/3$. This lattice (which is related to $SU(3)$ root lattice) admits a Z_3 rotation θ where $\theta v_1=v_2-v_1$ and $\theta v_2=-v_1$. There are now three fixed points $X=0,\left(v_1+v_2\right)/3, 2\left(v_1+v_2\right)/3$ modulo Γ . For this same example, we can also consider Z_6 rotation θ where $\theta v_1=v_2$ and $\theta v_2=v_2-v_1$. In this case, as one can verify, that the only fixed point is the origin modulo Γ , so there is only one fixed point.

我们来看几个例子。考虑 $T^2=R^2/\Gamma_2$ ，其中 Γ_2 是由两个向量 v_1 和 v_2 生成的二维格，取对 θ 的 Z_2 作用，即 $\theta X=-X$ ，其中满足 X � $R^2 . \theta X=X \bmod \Gamma$ 的向量被称为 θ 的不动点。不难看出本例中共有四个不动点，即 $X=0, v_1/2, v_2/2,\left(v_1+v_2\right)/2$ 。再看另一个例子: 设 v_1 与 v_2 长度相等且互相垂直 (即该格为正方格子)，那么旋转 Z_4 角度 $\pi/2$ 是这个格的一个自同构。具体来说，θ 作用为 $\left(v_1, v_2\right) \rightarrow\left(v_2,-v_1\right)$。此时方程 $\theta X=X \bmod \Gamma$ (模格向量) 仅有两个解，即 $X=0,\left(v_1+v_2\right)/2 \bmod \Gamma$ 。还有一个例子: 设 v_1 与 v_2 长度相等，二者夹角为 $\pi/3$。这个格 (与 $SU(3)$ 根格相关) 允许旋转 Z_3 变换 θ ，其中 $\theta v_1=v_2-v_1$ 且 $\theta v_2=-v_1$ 。此时模 Γ 共有三个不动点 $X=0,\left(v_1+v_2\right)/3, 2\left(v_1+v_2\right)/3$ 。对同一个例子，我们还可以考虑旋转 Z_6 变换 θ ，其中 $\theta v_1=v_2$ 且 $\theta v_2=v_2-v_1$ 。可以验证，这种情况下模 Γ 仅原点是不动点，因此只有一个不动点。

In fact there is a mathematical theorem which gives the number of fixed points as

事实上，存在一个数学定理给出不动点的数量为

$$
\text{Number of fixed points } =\det{}^{\prime}(1-\theta) \tag{11}
$$

where $\det{}^{\prime}$ means determinant over the non-zero eigenvalues of $(1-\theta)$. Indeed for the four examples given for T^2 , for Z_2 case the two eigenvalues of θ are $(-1,-1)$ so that $\det{}^{\prime}(1-\theta)=2 \times 2=4$, for Z_4 case the eigenvalues of θ are $\pm i$ so that $\det{}^{\prime}(1-\theta)=(1-i)(1+i)=2$, for Z_3 case the two eigenvalues of θ are $e^{\pm 2\pi i/3}$ so that $\det{}^{\prime}(1-\theta)=\left(1-e^{2\pi i/3}\right)\left(1-e^{-2\pi i/3}\right)=3$ and for Z_6 case two eigenvalues of θ are $e^{\pm 2\pi i/6}$ so that $\det{}^{\prime}(1-\theta)=\left(1-e^{2\pi i/6}\right)\left(1-e^{-2\pi i/6}\right)=1$.

其中 $\det'$ 表示 $(1-\theta)$ 非零本征值的行列式。对于给出的四个 T^2 实例确实如此：Z_2 情形下，θ 的两个本征值为 $(-1,-1)$，因此得到 $\det'(1-\theta)=2\times 2=4$；$Z_4$ 情形下，θ 的本征值为 $\pm i$，因此得到 $\det'(1-\theta)=(1-i)(1+i)=2$；$Z_3$ 情形下，θ 的两个本征值为 $e^{\pm 2\pi i/3}$，因此得到 $\det'(1-\theta)=\left(1-e^{2\pi i/3}\right)\left(1-e^{-2\pi i/3}\right)=3$；$Z_6$ 情形下，θ 的两个本征值为 $e^{\pm 2\pi i/6}$，因此得到 $\det'(1-\theta)=\left(1-e^{2\pi i/6}\right)\left(1-e^{-2\pi i/6}\right)=1$。

Now we can write down the mode expansion for X satisfying Eq. (10) in the T^2 examples discussed above. The eigenvalues of θ are $e^{\pm 2\pi i a/N}$ where N is the order of θ and a is some integer. Let z (a complex coordinate involving the two directions in T^2) be the eigenfunction with eigenvalue $e^{2\pi i a/N}$, then the mode expansion will be

现在我们可以为满足式 (10) 的 X 在上述讨论的 T^2 实例中写出模展开。θ 的本征值为 $e^{\pm 2\pi i a/N}$，其中 N 是 θ 的阶数，a 为某一整数。设 z (包含 T^2 中两个方向的复坐标) 是本征值为 $e^{2\pi i a/N}$ 的本征函数，则模展开可写为

$$z(\sigma,t)=z_0+\sum_{n\neq 0}\left(\alpha_{n+\frac{a}{N}}e^{i\left(n+\frac{a}{N}\right)(t+\sigma)}+\tilde{\alpha}_{n-\frac{a}{N}}e^{i\left(n-\frac{a}{N}\right)(t-\sigma)}\right) \tag{12}$$

Note that the modes now are shifted by $\frac{a}{N}$. This is because in the twisted sector $z(\sigma+2\pi,t)=e^{2\pi i a/N}z(\sigma,t)$ .z_0 in the above equation must be one of the fixed points. For each such fixed point, we can construct the Hilbert space of states by applying the creation operators $\alpha_{n+\frac{a}{N}}$ and $\tilde{\alpha}_{n-\frac{a}{N}}$ on the left and right moving part of the twisted sector ground state, where the ground state is defined by

注意现在模偏移了 $\frac{a}{N}$，这是因为在扭曲扇区中，上述方程里的 $z(\sigma+2\pi,t)=e^{2\pi i a/N}z(\sigma,t)$ .z_0 必须是不动点之一。对于每个这类不动点，我们可以通过在扭曲扇区基态的左行和右行部分上作用产生算符 $\alpha_{n+\frac{a}{N}}$ 与 $\tilde{\alpha}_{n-\frac{a}{N}}$ 来构造态的希尔伯特空间，其中基态满足

$$\alpha_{n+\frac{a}{N}}\mid 0>_L=0 \text{ for } n+\frac{a}{N}>0$$

$$\tilde{\alpha}_{n-\frac{a}{N}}\mid 0>_R=0 \text{ for } n-\frac{a}{N}>0 \tag{13}$$

Finally z_0 is a fixed point. For each such fixed point, there will be a Hilbert space of twisted states. Thus, there will be as many copies of such twisted Hilbert spaces as the number of fixed points.

最后，z_0 是一个不动点。对于每个这类不动点，都存在一个扭曲态的希尔伯特空间。因此，这类扭曲希尔伯特空间的份数等于不动点的个数。

The discussion above was just for T^2 . Now we include all the other directions as well as fermions. We start with the partition function for the 10 dimension theory compactified on T^6 (i.e., before orbifolding) in the light cone gauge [2]

以上讨论仅针对 T^2 。现在我们将所有其他方向以及费米子都包含进来。我们从光锥规范 [2] 下，紧致化在 T^6 上的 10 维理论 (即轨形化前) 的配分函数开始讨论

$$Z=\int\frac{d^2\tau}{\tau_2^3}Z_bZ_fZ_{\Gamma_{16}} \tag{14}$$

where the factors of τ_2 can be understood as follows. There are 4 non-compact directions so the integral over the continuous 4-momenta of $e^{-2\pi\tau_2p^2}$ gives $1/\tau_2^2$ and fixing the translational symmetry of world sheet torus gives a further $1/\tau_2$ as the volume of the world sheet torus is proportional to τ_2.Z_b is the partition function of the eight transverse X ' s (two of them non-compact and six of them compactified on some T^6 which for simplicity we will take as $\left(T^2\right)^3\Big)'$, Z_f is the (right moving) world sheet fermion partition function and $Z_{\Gamma_{16}}$ is the partition function of the left moving 16-dimension even self dual lattice.

其中 τ_2 的因子可以如下理解。有 4 个非紧致方向，因此对 $e^{-2\pi\tau_2p^2}$ 的连续 4 - 动量积分得到 $1/\tau_2^2$，固定世界面环面的平移对称性会进一步得到 $1/\tau_2$，因为世界面环面的体积与 τ_2.Z_b 成正比。X 是八个横向 X 的配分函数 (其中两个是非紧致的，六个在某个 T^6 上紧致化，为简单起见，我们将其取为 $\left(T^2\right)^3\Big)'$, Z_f)，$Z_{\Gamma_{16}}$ 是右行世界面费米子配分函数，$Z_{\Gamma_{16}}$ 是左行 16 维偶自对偶格的配分函数。

$$\begin{aligned}
Z_b&=\left|\frac{1}{\eta^8}\right|^2\sum_{(P_L,P_R)\in\gamma_{6,6}}q^{\frac{1}{2}P_L^2}\bar{q}^{\frac{1}{2}P_R^2}\\
Z_f&=\left(\sum_{p\in V_8}-\sum_{p\in Sp_8}\right)\frac{\bar{q}^{\frac{1}{2}p^2}}{\bar{\eta}^4}\\
Z_{\Gamma_{16}}&=\sum_{P\in\Gamma_{16}}\frac{q^{\frac{1}{2}P^2}}{\eta^{16}}
\end{aligned} \tag{15}$$

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[2] For simplicity we are taking torodial compactification with zero Wison lines but this can be relaxed.

[2] 为简化起见，我们考虑零 Wilson 线的环面紧致化，不过这一条件可以放宽。

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where Γ_{16} is 16 dimension. Even self dual lattice corresponding to $E_8\times E_8$ or $\mathrm{Spin}(32)/Z_2$ and $\gamma_{6,6}$ is an even self dual lattice with signature $(6,6)$ and consists of left moving and right moving momenta (P_L,P_R) made up of combinations of KK momenta and winding modes of T^6 . Note that for Z_f , as shown in Appendix C, we already used bosonization to write 8 (lightcone) world sheet fermions in terms of 4 world sheet bosons whose lattice momenta lie in $SO(8)$ Vector class in the NS sector (after GSO projection) and $SO(8)$ Spinor class in the R sector (again after GSO projection). Finally $\eta=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function which comes from the oscillator modes. Note that $q^{\frac{1}{24}}$ already takes care of the zero point shift in L_0 .

其中 Γ_{16} 是 16 维，对应 $E_8\times E_8$ 或 $\mathrm{Spin}(32)/Z_2$ 的偶自对偶晶格，$\gamma_{6,6}$ 是符号为 $(6,6)$ 的偶自对偶晶格，由左行动量和右行动量 (P_L,P_R) 构成，是 T^6 的 KK 动量与缠绕模的组合。注意对于 Z_f，如附录 C 所示，我们已经通过玻色化将 8 个 (光锥) 世界面费米子表示为 4 个世界面玻色子，其晶格动量属于 NS sector(GSO 投影后) 的 $SO(8)$ Vector 类，以及 R sector(同样经 GSO 投影后) 的 $SO(8)$ Spinor 类。最后 $\eta=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$ 是来自振荡模的戴德金 η 函数。注意 $q^{\frac{1}{24}}$ 已经处理了 L_0 中的零点移位问题。

Consider an orbifold group generated by $g \in Z_N$ which rotates the three T^2 -planes by $2\pi\,(r_1/N, r_2/N, r_3/N)$ with r_i being integers. We call it symmetric orbifold because g is acting symmetrically on the both the left and right moving sectors of the three T^2 . Of course it is understood that this rotation is a symmetry of the lattice describing the three 2-tori. Since the world sheet supercurrent $T_F = \bar{\partial} X^i \psi_i$ must be invariant under the orbifold action, on the world sheet fermions the action of g is fixed once we specify its action on X’s (see Appendix C for details).

考虑一个由 $g \in Z_N$ 生成的 orbifold 群，它将三个 T^2 平面以 $2\pi\,(r_1/N, r_2/N, r_3/N)$ 旋转，其中 r_i 为整数。我们称之为对称 orbifold，因为 g 对三个 T^2 的左行和右行扇区作用对称。当然，这个旋转是描述三个二维环面晶格的一个对称性。由于世界面超流 $T_F = \bar{\partial} X^i \psi_i$ 必须在 orbifold 作用下不变，一旦我们指定了 g 对 X 的作用，它对世界面费米子的作用就确定了 (详见附录 C)。

$$g\,|p >= e^{2\pi i p\cdot v_f}\,|\,p > \tag{16}$$

where

其中

$$v_f = (0, r_1/N, r_2/N, r_3/N) \tag{17}$$

(the first zero just means that g action on the world sheet fermions along the two light cone directions in R^4 is identity). Note that for Sp_8 weights, $e^{2\pi i p.v_f} = e^{\pi i(\pm r_1 + \pm r_2 \pm r_3)/N}$ and therefore for this action to be Z_N we have the requirement

(第一个零仅表示 g 对 R^4 中两个光锥方向上的世界面费米子作用为恒等)。注意对 Sp_8 权，$e^{2\pi i p.v_f} = e^{\pi i(\pm r_1 + \pm r_2 \pm r_3)/N}$，因此要让这个作用满足 Z_N，我们要求

$$\sum_i r_i = \text{ even} \tag{18}$$

and furthermore for some supersymmetry to be unbroken we must have $\pm r_1 \pm r_2 \pm r_3$ should be 0 mod $2N$ for some choice of signs.

此外，为了让部分超对称不被破缺，我们必须要求对某些符号取法而言，$\pm r_1 \pm r_2 \pm r_3$ 模 $2N$ 等于 0。

Furthermore we can also let g to act on the Γ_{16} lattice states $g\,|P >= e^{2\pi i P.V}|\,P >$ for some 16=dimension vector V such that $NV \in \Gamma_{16}$ (this ensures that $g^N \mid P >= \; |P >$) .

此外我们还可以令 g 作用在 Γ_{16} 晶格态 $g\,|P >= e^{2\pi i P.V}|P >$ 上，作用对象满足 16 维矢量 V 使得 $NV \in \Gamma_{16}$ (这保证了 $g^N \mid P >= \; |P >$)。

The partition function in $(1, g)$ sector is $\mathrm{Tr} g\, q^{L_0} \bar{q}^{\bar{L}_0}$. We note that states carrying non-zero $|p >=|\,(P_L, P_R) >\in \gamma_{(6,6)}$ vanish in this trace. To see this, consider the states $]s >= \sum_{r=1}^{N} e^{-2\pi i r s/N}\;|g^r p >$ for $s = 0, \ldots, N-1$. Then $g|\,s >=$

$\sum_{r=1}^{N} e^{-2\pi irs/N}\left|g^{r+1}p>=\sum_{r=1}^{N} e^{-2\pi i(r-1)s/N}\right|g^r p>=e^{2\pi is/N}|s>$. Thus states $|s>$ are eigenstates of g with eigenvalues $e^{2\pi is/N}$ for $s=0,1,\ldots,N-1$. Since $(g^r P_L)^2=P_L^2$ and $(g^r P_R)^2=P_R^2, L_0$ and $\bar{L}_0$ values for all the states $|s>$ are the same. As a result summing over all s, $\mathrm{Tr}\, gq^{L_0}\bar{q}^{\bar{L}_0}$ vanishes. This implies that in $(1,g)$ sector only $(P_L,P_R)=(0,0)$ survives.

$(1,g)$ 扇区的配分函数为 $\mathrm{Tr} g\, q^{L_0}\bar{q}^{\bar{L}_0}$ 。我们注意到，携带非零 $|p>=|(P_L,P_R)>\in\gamma_{(6,6)}$ 的态在该迹中为零。要理解这一点，考虑态 $]s>=\sum_{r=1}^{N} e^{-2\pi irs/N}|g^r p>$ for $s=0,\ldots,N-1$. Then $g|s>=\sum_{r=1}^{N} e^{-2\pi irs/N}\left|g^{r+1}p>=\sum_{r=1}^{N} e^{-2\pi i(r-1)s/N}\right|g^r p>=e^{2\pi is/N}|s>$: 因此态 $|s>$ 是 g 的本征态，对 $s=0,1,\ldots,N-1$ 而言本征值为 $e^{2\pi is/N}$ 。由于 $(g^r P_L)^2=P_L^2$ 、$(g^r P_R)^2=P_R^2, L_0$ ，所有态 $|s>$ 的 $\bar{L}_0$ 值都相同，因此对所有 s, $\mathrm{Tr}\, gq^{L_0}\bar{q}^{\bar{L}_0}$ 求和后结果为零。这说明在 $(1,g)$ 扇区中只有 $(P_L,P_R)=(0,0)$ 保留下来。

More generally, when the $\Gamma_{22,6}$ lattice is not factorized in terms of $\Gamma_{16,0}$ and 3 $\Gamma_{2,2}$, the above argument shows that in $(1,g)$ sector, the states with $\theta P\neq P$ cancel out in the trace and hence only states with $\theta P=P$ survive. The set of all such states will be a sublattice of $\Gamma_{22,6}$ which are invariant under θ . We denote this invariant sublattice by I . Clearly I is an even and integral lattice.

更一般地，当 $\Gamma_{22,6}$ 晶格无法分解为 $\Gamma_{16,0}$ 和 3 个 $\Gamma_{2,2}$ 时，上述论证表明，在 $(1,g)$ 扇区中，带有 $\theta P\neq P$ 的态会在迹中抵消，因此只有带有 $\theta P=P$ 的态保留下来。所有这类态的集合构成 $\Gamma_{22,6}$ 在 θ 下不变的子晶格，我们将这个不变子晶格记为 I 。显然 I 是偶整晶格。

It follows that

由此可得

$$Z_b^{1,g}=\left|\frac{1}{\eta^2}\prod_{j=1}^{3}\frac{1}{q^{\frac{1}{12}}\prod_{n=1}^{\infty}\left(1-q^n e^{2\pi ir_j/N}\right)\left(1-q^n e^{-2\pi ir_j/N}\right)}\right|^2$$
$$=\left|\frac{1}{\eta(\tau)^2}\prod_{j=1}^{3}\frac{2\sin(\pi r_i/N)\,\eta(\tau)}{\theta_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(\frac{r_i}{N},\tau\right)}\right|^2 \tag{19}$$

where product form of $\theta_{\left(\frac{1}{2},\frac{1}{2}\right)}$ has been used. In the above expression we have assumed that none of the r_i are trivial, i.e., 0 mod N . If one of the r_i (say r_3) is trivial, then in the second line of above equation product over j will go from 1 to 2 and $\eta(\tau)^2$ will be replaced by $\eta(\tau)^4$.

这里我们使用了 $\theta_{\left(\frac{1}{2},\frac{1}{2}\right)}$ 的乘积形式。在上式中我们假定所有 r_i 都非平庸，即满足 0 mod N 。如果其中一个 r_i (即 r_3) 是平庸的，那么上述方程第二行中对 j 的乘积将遍历 1 到 2，且 $\eta(\tau)^2$ 会被替换为 $\eta(\tau)^4$ 。

$$Z_f^{1,g}=\left(\sum_{p\in V_8}-\sum_{p\in Sp_8}\right)\frac{\bar{q}^{\frac{1}{2}p^2}e^{2\pi ip\cdot v_f}}{\bar{\eta}^4} \tag{20}$$

$$Z_{\Gamma_{16}}^{1,g} = \frac{\sum_{P\in\Gamma_{16}} q^{\frac{1}{2}P^2} e^{2\pi i P.V}}{\eta^{16}} \tag{21}$$

(g,1) Sector

(g,1) 扇区

The expressions for $(1, g)$ sector above were obtained by applying g on the Hilbert space of the original toroidal theory and taking trace. This amounts to computing the path integral over the world sheet fields with boundary conditions along the two

上文 $(1, g)$ 扇区的表达式是通过对原环面理论的希尔伯特空间作用 g 并求迹得到的。这相当于计算世界面上沿两个方向满足边界条件的场的路径积分

1-cycles on the world sheet torus being $X(z+1) = X(z)$ and $X(z+\tau) = gX(z)$. By exchange of the two 1-cycles on the world sheet torus we will get the result for $(g, 1)$ sector. But this exchange of the two 1-cycles on the world sheet torus amounts to the S -transformation $\tau \to -1/\tau$. Thus $(g, 1)$ sector can be obtained by $\tau \to \tau' = -1/\tau$ transformation applied on $(1, g)$ sector. This involves doing a Poisson resummation of the lattice part using the equations given in the Appendix A and doing the Poisson resummation of the lattice sums in Eqs. (6) and (7).

世界面环面上的两个 1-周期分别为 $X(z+1) = X(z)$ 和 $X(z+\tau) = gX(z)$ 。交换世界面环面上的两个 1-周期，我们就能得到 $(g, 1)$ 扇区的结果。而交换这两个 1-周期等价于 S 变换 $\tau \to -1/\tau$ 。因此 $(g, 1)$ 扇区可以通过对 $(1, g)$ 扇区作用 $\tau \to \tau' = -1/\tau$ 变换得到，这需要利用附录 A 给出的公式对晶格部分做泊松重求和，再对式 (6) 和 (7) 中的晶格和做泊松重求和。

Let us start with Z_b .

我们从 Z_b 开始讨论。

$$\theta_{\frac{1}{2},\frac{1}{2}}\left(\frac{r_i}{N},\tau\right) = \sqrt{-i\tau'}e^{\pi i\left(\frac{r_i}{N}\right)^2\tau'} i\theta_{\frac{1}{2},-\frac{1}{2}}\left(\frac{r_i}{N}\tau',\tau'\right)$$

$$= -i\sqrt{-i\tau'}e^{\pi i\left(\frac{r_i}{N}\right)^2\tau'}\theta_{\frac{1}{2},\frac{1}{2}}\left(\frac{r_i}{N}\tau',\tau'\right)$$

$$= -i\sqrt{-i\tau'}e^{\pi i\left(\frac{r_i}{N}\right)^2\tau'}\left(ie^{\pi i\left(\frac{r_i}{N}\right)\tau'}\right)$$

$$q'^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q'^n)\left(1-e^{2\pi i\frac{r_i}{N}\tau'}q'^n\right)\left(1-e^{-2\pi i\frac{r_i}{N}\tau'}q'^{n-1}\right)$$

$$= \sqrt{-i\tau'}q'^{\frac{1}{2}\left(\left(\frac{r_i}{N}\right)^2+\frac{r_i}{N}\right)}q'^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q'^n)\left(1-q'^{n+\frac{r_i}{N}}\right)\left(1-q'^{n-1-\frac{r_i}{N}}\right)$$

$$= \sqrt{-i\tau'}\eta(\tau')\, q'^{\frac{1}{2}\left(\left(\frac{r_i}{N}\right)^2+\frac{r_i}{N}\right)} q'^{\frac{1}{12}} \prod_{n=1}^{\infty}\left(1-q'^{n+\frac{r_i}{N}}\right)\left(1-q'^{n-1-\frac{r_i}{N}}\right)$$

(22)

where $\tau' = -1/\tau$ and $q' = e^{2\pi i\tau'}$. In the first line above we have used (44) in the second line (39) with $(m_1, m_2) = (0, -2\beta)$, in the third line (42) and in the fifth line used the definition of η . Now there are two cases (i) $0 < r_i < N$ and (ii) $-N < r_i < 0$. In case (ii), the exponents of q' inside the product over n are always positive, i.e., $n + \frac{r_i}{N}$ and $n - 1 - \frac{r_i}{N}$ are positive for $n = 1$ to infinity and therefore can be written as $\prod_{n=1}^{\infty}\left(1-q'^{n-\left|\frac{r_i}{N}\right|}\right)\prod_{n=0}^{\infty}\left(1-q'^{n+\left|\frac{r_i}{N}\right|}\right)$. This has the correct interpretation of energy levels coming from applying the twisted oscillator creation operators of z_i and $\bar{z}_i$. In case (i), however, the last factor above namely $\left(1-q'^{n-1-\frac{r_i}{N}}\right)$ for $n = 1$ becomes $\left(1-q'^{-\frac{r_i}{N}}\right)$, i.e., the power of q' becomes negative which cannot be interpreted as the energy level coming from applying a creation oscillator operator. So we rewrite it as $-q'^{-\frac{r_i}{N}}\left(1-q'^{\frac{r_i}{N}}\right)$ and now $\prod_{n=1}^{\infty}(1- q'^{n+\frac{r_i}{N}})\left(1-q'^{n-1-\frac{r_i}{N}}\right)$ becomes $-q'^{-\frac{r_i}{N}}\prod_{n=0}^{\infty}\left(1-q'^{n+\frac{r_i}{N}}\right)\prod_{n=1}^{\infty}\left(1-q'^{n-\frac{r_i}{N}}\right)$ and again the product has the interpretation of energy levels coming from applying the twisted oscillator creation operators of z_i and $\bar{z}_i$. Both these cases can be combined into the formula:

其中 $\tau' = -1/\tau$ 和 $q' = e^{2\pi i\tau'}$ 。上文中第一处使用了式 (44)，第二处结合 $(m_1, m_2) = (0, -2\beta)$ 使用了式 (39)，第三处使用了式 (42)，第五处使用了 η 的定义。现在分为两种情况:(i) $0 < r_i < N$ ，(ii) $-N < r_i < 0$。在情况 (ii) 中，乘积遍历 n 时，q' 的指数始终为正，即对于 $n = 1$ 到无穷大，$n + \frac{r_i}{N}$ 和 $n-1-\frac{r_i}{N}$ 均为正,因此可写为 $\prod_{n=1}^{\infty}\left(1-q'^{n-\left|\frac{r_i}{N}\right|}\right)\prod_{n=0}^{\infty}\left(1-q'^{n+\left|\frac{r_i}{N}\right|}\right)$。这恰好对应施加 z_i 和 $\bar{z}_i$ 的扭振子产生算符得到的能级,物理解释正确。但在情况 (i) 中,上述最后一个因子即对应 $n = 1$ 的 $\left(1-q'^{n-1-\frac{r_i}{N}}\right)$ 变为 $\left(1-q'^{-\frac{r_i}{N}}\right)$,即 q' 的幂次变为负数,无法解释为施加产生算符得到的能级。因此我们将其重写为 $-q'^{-\frac{r_i}{N}}\left(1-q'^{\frac{r_i}{N}}\right)$，此时 $\prod_{n=1}^{\infty}(1- q'^{n+\frac{r_i}{N}})\left(1-q'^{n-1-\frac{r_i}{N}}\right)$ 变为 $-q'^{-\frac{r_i}{N}}\prod_{n=0}^{\infty}\left(1-q'^{n+\frac{r_i}{N}}\right)\prod_{n=1}^{\infty}\left(1-q'^{n-\frac{r_i}{N}}\right)$，该乘积也可以解释为施加 z_i 和 $\bar{z}_i$ 的扭振子产生算符得到的能级。两种情况可以合并为公式:

$$\theta_{\frac{1}{2},\frac{1}{2}}\left(\frac{r_i}{N},\tau\right) = (-\operatorname{sign}(r_i))\sqrt{-i\tau'}\eta(\tau')\, q'^{\frac{1}{2}\left(\left(\frac{r_i}{N}\right)^2-\left|\frac{r_i}{N}\right|\right)}$$

$$q'^{\frac{1}{12}}\prod_{n=0}^{\infty}\left(1-q'^{n+\left|\frac{r_i}{N}\right|}\right)\prod_{n=1}^{\infty}\left(1-q'^{n--\left|\frac{r_i}{N}\right|}\right) \quad (23)$$

This result is not surprising. Indeed, on the right hand side, everything else except $\operatorname{sign}(r_i)$ depends only on the modulus of r_i and the left hand is odd function of r_i .

这一结果并不意外。事实上，右侧除 $\operatorname{sign}(r_i)$ 外的所有项仅依赖于 r_i 的模，而左侧是 r_i 的奇函数。

Including also Eq. (46) namely $\eta(\tau) = \sqrt{-i\tau'}\eta(\tau')$ as well as $\tau_2 = \frac{\tau'_2}{\tau'\bar{\tau}'}$ and $d^2\tau = \frac{d^2\tau'}{\left(\tau'\bar{\tau}'\right)^2}$ we conclude from Eq. (19) that

再纳入式 (46)，即 $\eta(\tau) = \sqrt{-i\tau'}\eta(\tau')$，以及 $\tau_2 = \frac{\tau'_2}{\tau'\bar{\tau}'}$ 和 $d^2\tau = \frac{d^2\tau'}{\left(\tau'\bar{\tau}'\right)^2}$，我们由式 (19) 得出结论:

$$\frac{d^2\tau}{\tau_2^3} Z_b^{(1,g)}(\tau,\bar{\tau}) = \frac{d^2\tau'}{\tau'^3} \det'(1-g) \left|\frac{1}{\eta(\tau')^2}\right|$$

$$\prod_{j=1}^{3} \frac{q'^{-\frac{1}{12}+\frac{1}{2}\frac{|r_j|}{N}\left(1-\frac{|r_j|}{N}\right)}}{\prod_{n=0}^{\infty}\left(1-q'^{n+\left|\frac{r_j}{N}\right|}\right)\prod_{n=1}^{\infty}\left(1-q'^{n-\left|\frac{r_j}{N}\right|}\right)^2}$$

$$= \frac{d^2\tau'}{\tau'^3_2} Z_b^{(g,1)}(\tau,\bar{\tau}) \tag{24}$$

where we have used the following

此处我们用到了如下关系

$$det'(1-g) = \prod_{j=1}^{3}\left(\left(1-e^{2\pi i\frac{r_j}{N}}\right)\left(1-e^{-2\pi i\frac{r_j}{N}}\right)\right) = \prod_{j=1}^{3}\left(2\sin\left(\pi\frac{r_j}{N}\right)\right)^2 \tag{25}$$

This is just the number of fixed points. The right hand side clearly is the partition function of the twisted variables $(z_j,\bar{z}_j)$ which satisfy the boundary condition $(z_j(\sigma+2\pi), \bar{z}_j(\sigma+2\pi)) = \left(e^{2\pi i\frac{r_j}{N}} z_j(\sigma), e^{-2\pi i\frac{r_j}{N}} \bar{z}_j(\sigma)\right)$ and therefore their modes are shifted accordingly which is reflected in the fact that powers of q' are integers shifted by $\pm\frac{r_j}{N}$.

这正是不动点的数量。右侧显然是扭曲变量 $(z_j,\bar{z}_j)$ 的配分函数，这些变量满足边界条件 $(z_j(\sigma+2\pi), \bar{z}_j(\sigma+2\pi)) = \left(e^{2\pi i\frac{r_j}{N}} z_j(\sigma), e^{-2\pi i\frac{r_j}{N}} \bar{z}_j(\sigma)\right)$ ，因此它们的模相应发生偏移，这体现为 q' 的幂次是被 $\pm\frac{r_j}{N}$ 偏移后的整数。

Finally the factor $q'^{-\frac{1}{12}+\frac{1}{2}\frac{|r_j|}{N}\left(1-\frac{|r_j|}{N}\right)}$ in Eq. (24) can be understood as follows. Recall that for the usual untwisted boson (i.e., $r_j = 0$) this power just comes from zero point shift $-1/24$ for each boson and therefore for a pair of bosons (that is for a plane) it is $-1/12$, which in turn can be understood as coming from normal ordering (i.e., moving the annihilation operators in L_0 to the right). This gives $\frac{1}{2}\sum_{n=1}^{\infty} n = -1/24$ using zeta function regularization. In the twisted sector since the modes are shifted by r_j/N , the resulting normal ordering gives (for a plane) $\frac{1}{2}\left(\sum_{n=0}^{\infty}\left(n+\left|\frac{r_j}{N}\right|\right)+\sum_{n=1}^{\infty}\left(n-\left|\frac{r_j}{N}\right|\right)\right)$ which again using zeta function (actually using Hurwitz zeta function), which in turn for our case reduces to $-\frac{1}{4}(B_2(a)+B_2(1-a))$ where $a = +\left|\frac{r_j}{N}\right|$ and B is the Bernoulli polynomial and $B_2(a) = a^2 - a + \frac{1}{6}$. Substituting $a = |r_j/N|$ we get the result.

最后，式 (24) 中的因子 $q'^{-\frac{1}{12}+\frac{1}{2}\frac{|r_j|}{N}\left(1-\frac{|r_j|}{N}\right)}$ 可以按如下方式理解。我们知道，对于常规非扭曲玻色子 (即 $r_j = 0$)，该幂次恰好来自每个玻色子的零点偏移 $-1/24$ ，因此对于一对玻色子 (即对应一个平面)，结果为 $-1/12$, 这又可以从正规序 (即将 L_0 中的湮灭算符移到右侧) 得到解释。利用 ζ 函数正规化，该过程给出 $\frac{1}{2}\sum_{n=1}^{\infty} n = -1/24$ 。在扭曲扇区中，由于模被 r_j/N 偏移，最终正规序给出 (一个平面的结果) $\frac{1}{2}\left(\sum_{n=0}^{\infty}\left(n+\left|\frac{r_j}{N}\right|\right)+\sum_{n=1}^{\infty}\left(n-\left|\frac{r_j}{N}\right|\right)\right)$, 再次利用 ζ 函数 (实际是赫尔维茨 ζ 函数), 在我们的问题中它可以约化为 $-\frac{1}{4}\left(B_2(a)+B_2(1-a)\right)$, 其中 $a = +\left|\frac{r_j}{N}\right|$, B 是伯努利多项式，且 $B_2(a) = a^2 - a + \frac{1}{6}$ 。代入 $a = |r_j/N|$ 后我们得到了该结果。

The remaining pieces- the right moving fermionic and left moving gauge sector, can be rewritten as functions of τ' by Poisson resummation. It is useful to do this for a general even and integral lattice I (even means that length square of every vector in I is even and integral means that dot product of any two vectors in I is integer). This of course implies that I is a sublattice of its dual lattice I^* . Let $v \in I^*$ and let w be a vector such that $Nw \in I$. This has been done in Appendix A with the result given in Eq. (36). Applying this for $Z_f^{1,g}$, I is $SO(8)$ root lattice and v is $SO(8)$ vector and spinor weights with relative negative sign. In this case I is Sc_8 class and Vol = Sqrt $|I^*/I| = 2$ and $d = 4$

其余部分——右动费米子和左动规范规范，可以通过泊松求和重写为 τ' 的函数。对于一般的偶整格 I (偶指 I 中所有向量的长度平方均为偶数，整指 I 中任意两个向量的点积均为整数)，这样处理是很方便的。这自然说明 I 是其对偶格 I^* 的子格。设 $v \in I^*$ ，令 w 为满足 $Nw \in I$ 的向量。这部分已在附录 A 中完成，结果见式 (36)。将其应用于 $Z_f^{1,g}$, I 是 $SO(8)$ 根格，v 是带相对负号的 $SO(8)$ 向量和旋量权。在此情形下，I 是 Sc_8 类，体积 Vol = 等于根号 Sqrt $|I^*/I| = 2$ 即 $d = 4$

$$Z_f^{1,g}(\bar{\tau}) = \frac{1}{2}\frac{1}{\bar{\eta}\left(\bar{\tau}'\right)^4}\sum_{v'\in I^*/I}\sum_{P\in I}\bar{q}'^{\frac{1}{2}\left(P+v'+v_f\right)^2}\left(e^{2\pi i V_8.v'} - e^{2\pi i Sp_8.v'}\right) \tag{26}$$

where v' is a representative of each of the four classes Sc_8, V_8, Sp_8 , and Sp_8' . Using the fact that Sc_8 dotted with all classes is integer and $Sp_8'.V_8$ as well as $Sp_8'.Sp_8$ is half mod integer, we see that $v' \in Sc_8$ and $v' \in Sp_8'$ vanishes. When $v' \in V_8$, $\left(e^{2\pi i V_8.v'} - e^{2\pi i Sp_8.v'}\right) = 1 + 1 = 2$ and when $v' \in Sp_8$, $\left(e^{2\pi i V_8.v'} - e^{2\pi i Sp_8.v'}\right) = -1 - 1 = -2$. The resulting factor 2 cancels with the prefactor 1/2 in the above equation and the final result for the fermionic part is

其中 v' 是四个类 Sc_8, V_8, Sp_8 和 Sp_8' 中每个类的代表元。利用 Sc_8 与所有类的点积均为整数、$Sp_8'.V_8$ 和 $Sp_8'.Sp_8$ 同为半整数模 1，我们可知 $v' \in Sc_8$ 和 $v' \in Sp_8'$ 等于零。当 $v' \in V_8$ 时得到 $\left(e^{2\pi i V_8.v'} - e^{2\pi i Sp_8.v'}\right) = 1 + 1 = 2$ ，当 $v' \in Sp_8$, $\left(e^{2\pi i V_8.v'} - e^{2\pi i Sp_8.v'}\right) = -1 - 1 = -2$ 时得到对应结果。得到的因子 2 与上述方程中的前置因子 1/2 抵消，费米子部分的最终结果为

$$Z_f^{1,g}(\bar{\tau}) = \frac{1}{\bar{\eta}\left(\bar{\tau}'\right)^4}\left(\sum_{P\in V_8} - \sum_{P\in Sp_8}\right)\bar{q}'^{\frac{1}{2}\left(P+v_f\right)^2} = Z_f^{g,1}\left(\bar{\tau}'\right) \tag{27}$$

Finally using Eq. (36), the Γ_{16} part of the partition function Eq. (8) becomes

最后利用式 (36)，配分函数 (8) 中的 Γ_{16} 部分变为

$$Z_{\Gamma_{16}}^{1,g}(\tau)=\frac{\sum\limits_{P\in\Gamma_{16}} q'^{\frac{1}{2}(P+V)^2}}{\eta(\tau')^{16}}=Z_{\Gamma_{16}}^{g,1}(\tau') \tag{28}$$

Level Matching Condition

能级匹配条件

This condition comes from requiring that under $\tau' \to \tau'+N$, the $(g,1)$ sector should go to $\left(g,g^N\right)$ sector but this must be the same as $(g,1)$ sector since g^N should be identity. Clearly $Z_b^{g,1}$ is invariant under $\tau' \to \tau'+N$ as can be seen from Eq. (11) as it is a modsquare. From $Z_f^{g,1}$ and $Z_{\Gamma_{16}}^{g,1}$, we find the condition

该条件源自下述要求: 在变换 $\tau' \to \tau'+N$ 下，$(g,1)$ 区应变为 $\left(g,g^N\right)$ 区，但由于 g^N 应为恒等变换，所得结果必须与原 $(g,1)$ 区相同。显然 $Z_b^{g,1}$ 在 $\tau' \to \tau'+N$ 变换下不变，这一点可从式 (11) 看出，因为它是模平方。结合 $Z_f^{g,1}$ 与 $Z_{\Gamma_{16}}^{g,1}$ ，我们得到条件

$$\frac{N}{2}\left(v_f^2-V^2\right)\in Z \tag{29}$$

where we have used Eq. (18) which says that $Np.v_f$ is integer for all p in V_8 and Sp_8 and also we have required $NV\in\Gamma_{16}$ and therefore $NP.V$ is an integer for all $P\in\Gamma_{16}$

此处我们用到了式 (18): 式 (18) 表明对于 V_8 和 Sp_8 中的所有 p ，$Np.v_f$ 均为整数；同时我们要求 $NV\in\Gamma_{16}$ 为整数，因此对于所有 $P\in\Gamma_{16}$ ，$NP.V$ 也必为整数

Example- Z_3 Orbifold

例题: Z_3 轨形

We take $\left(T^2\right)^3/Z_3$ where Z_3 acts on each of the three T^2 as rotation by $2\pi/3$. This of course means that each of the T^2 are described by lattices that have Z_3 automorphism. $v_f=\left(\frac{1}{3},\frac{1}{3},-\frac{2}{3}\right)$. Note that $Nv_f^2/2=3/2\,(1/9+1/9+4/9)=1$ and therefore level matching condition is already satisfied and we do not need any V . However to make the model more interesting, let us choose V as follows. Decompose one of the E_8 in terms of $SU(3)\times E_6.E_8$ lattice then splits as (R_3,R_6) , (F_3,F_6) , and $\left(\bar{F}_3,\bar{F}_6\right)$ where R_3,F_3 , and $\bar{F}_3$ are, respectively, $SU(3)$ root lattice, fundamental weights, i.e.,(3) and anti-fundamental weights, i.e., $\left(\overline{3}\right)$ shifted by the $SU(3)$ root lattice. Similarly R_6,F_6 , and $\bar{F}_6$ are, respectively, E_6 root lattice, fundamental weights, i.e., (27) and anti-fundamental weights, i.e., (27) shifted by the E_6 root lattice. In particular the 240 roots of E_8 are given by 6SU(3) roots +72 roots of E_6 and (3, 27) and $\left(\overline{3},\overline{27}\right)$. Now we choose V to be the $SU(3)$ anti-fundamental weight $\bar{F}_3$. Clearly the phase $e^{2\pi iP.V}$ gives 1 when P is in (R_3,R_6) and second E_8 , the phase is $e^{2\pi i/3}$ for all $P\in(F_3,F_6)$ and $e^{-2\pi i/3}$ for all $P\in\left(\bar{F}_3,\bar{F}_6\right)$

我们取 $(T^2)^3/Z_3$，其中 Z_3 对三个 T^2 中的每一个都作用为转动 $2\pi/3$。这说明每个 T^2 都由具有 Z_3 自同构的格描述。$v_f = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$。注意 $Nv_f^2/2 = 3/2\,(1/9 + 1/9 + 4/9) = 1$，因此层匹配条件已经满足，我们不需要额外的 V。不过为了让模型更有意思，我们按如下方式选择 V。将其中一个 E_8 按 $SU(3) \times E_6.E_8$ 格分解，该格可拆分为 (R_3, R_6)、(F_3, F_6) 和 $(\bar{F}_3, \bar{F}_6)$，其中 R_3, F_3 和 $\bar{F}_3$ 分别是 $SU(3)$ 根格、基本权 (即 (3)) 和反基本权，也就是被 $SU(3)$ 根格平移了的 $(\overline{3})$。类似地，R_6, F_6 和 $\bar{F}_6$ 分别是 E_6 根格、基本权 (即 (27)) 和反基本权，也就是被 E_6 根格平移了的 (27)。特别地，E_8 的 240 个根由 6SU (3) 根加上 E_6 的 72 个根，再加上 (3, 27) 和 $(\overline{3}, \overline{27})$ 给出。现在我们选取 V 为 $SU(3)$ 反基本权 $\bar{F}_3$。显然，当 P 属于 (R_3, R_6) 时相位 $e^{2\pi i P.V}$ 取值为 1；对于第二个 E_8，所有 $P \in (F_3, F_6)$ 对应的相位为 $e^{2\pi i/3}$，所有 $P \in (\bar{F}_3, \bar{F}_6)$ 对应的相位为 $e^{-2\pi i/3}$

Let us see what are the massless states. We will focus on the bosonic states (the fermionic states will be just their superpartners). In the untwisted sector these are just the original 10-dimension massless fields that are g -invariant. These are $G_{\mu\nu}$, $G_{i\bar{j}}$ (a short hand notation for $G_{z_i,\bar{z}_j}$ where z_i for $i = 1, 2, 3$ are the complex coordinates on the three T^2 such that under the Z_3 rotation $z_i \to e^{2\pi i/3} z_i$), $B_{\mu\upsilon}$ (which is axion via duality $dB{=}^{*}d\chi$), $B_{i\bar{j}}$. Finally the gauge fields A_μ are in $SU(3) \times E_6 \times E_8$ (in other words the first E_8 where we have put V is broken to $SU(3) \times E_6$) and finally $3A_i$ in $(\overline{3}, \overline{27}, 1)$ and $3A_{\bar{i}}$ in $(3, 27, 1)$ of $SU(3) \times E_6 \times E_8$.

我们来看一下哪些是零质量态。我们将重点讨论玻色子态 (费米子态恰好是它们的超对称伙伴)。在 untwisted sector(无扭曲扇区) 中，这些态就是原本的十维零质量场中满足 g 不变性的场。它们包括 $G_{\mu\nu}$、$G_{i\bar{j}}$ (是 $G_{z_i,\bar{z}_j}$ 的简写记法，其中三个复坐标都取 $i = 1, 2, 3$ 时，满足 z_i，这些复坐标定义在三个 T^2 上，在 Z_3 转动下满足 $z_i \to e^{2\pi i/3} z_i$)、$B_{\mu\upsilon}$ (通过对偶性 $dB{=}^{*}d\chi$ 可知它是轴子)、$B_{i\bar{j}}$。最后，规范场 A_μ 属于 $SU(3) \times E_6 \times E_8$ (换句话说，我们原本嵌入了 V 的初始 E_8 会破缺到 $SU(3) \times E_6$)，最终得到 $3A_i$ 处于 $SU(3) \times E_6 \times E_8$ 的 $(\overline{3}, \overline{27}, 1)$，$3A_{\bar{i}}$ 处于 $SU(3) \times E_6 \times E_8$ 的 $(3, 27, 1)$。

In the twisted sector, in the right moving sector the zero point shift in energy is from Eq. (24), $\sum_i \frac{1}{2} \frac{|r_i|}{N} \left(1 - \frac{|r_i|}{N}\right) = 3\frac{1}{2}\frac{1}{3}\frac{2}{3} = \frac{1}{3}$ from bosons and from Eq. (27) the minimum energy appears for $P = (0, 0, 0, 1)$ so that $(P + v_f) = -\left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and is $\frac{1}{2} 3 \left(\frac{1}{3}\right)^2 = 1/6$. Thus the total is $1/6 + 1/3 = 1/2$ which cancels the zero point energy $-1/2$ giving rise to right moving $\bar{L}_0 = 0$. This means that the left moving L_0 must be also zero for the corresponding physical states. Taking into account the zero point shift 1/3 coming from left moving bosons (from Eq. (11)) we have the following possibilities from Γ_{16} as V is fundamental of $SU(3)$ in the decomposition of first E_8 in terms of $SU(3) \times E_6$: (1) $P + V$ is weight of 27 of E_6 whose half the length square is 2/3 and (2) we can choose $P + V$ as the weight of 3 of $SU(3)$ whose half length square is 1/3 and excite a 1/3 oscillator mode from Eq. (11) the multiplicity of which is 3 . Thus the massless states from each fixed point in g -twisted sector are $(1, 27, 1)$ and 3 copies of $(3, 1, 1)$. Since there are 27 fixed points $\left(\det{}'(1 - g) = 27\right)$ we have 27 copies of $(1, 27, 1)$ and 81 copies of $(3, 1, 1)$ from g -twisted sector. Similarly from g^2 twisted sector we get 27 copies of $(1, 27, 1)$ and 81 copies of $(\overline{3}, 1, 1)$.

在扭曲扇区中，右行扇区的零点能量移由式 (24) 给出: 玻色子贡献 $\sum_i \frac{1}{2}\frac{|r_i|}{N}\left(1-\frac{|r_i|}{N}\right) = 3\frac{1}{2}\frac{1}{3}\frac{2}{3} = \frac{1}{3}$；根据式 (27)，$P = (0,0,0,1)$ 对应最低能量，因此满足 $\left(P+v_f\right) = -\left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$，能量为 $\frac{1}{2}3\left(\frac{1}{3}\right)^2 = 1/6$。因此总能量移为 $1/6+1/3 = 1/2$，恰好抵消零点能量 $-1/2$，得到右行 $\bar{L}_0 = 0$。这意味着对应物理态的左行 L_0 也必须为零。考虑左行玻色子带来的零点能量移 1/3 (来自式 (11))，将首个 E_8 按 $SU(3)\times E_6$ 分解后，$SU(3)$ 的基础表示为 V，我们从 Γ_{16} 可得以下两种情况:(1) $P+V$ 是 E_6 的 27 维表示权，其半长度平方为 2/3；(2) 可令 $P+V$ 为 $SU(3)$ 的 3 维表示权，其半长度平方为 1/3，再激发一个来自式 (11) 的 1/3 振子模，该模的简并度为 3。因此，g 扭曲扇区中每个不动点给出的零质量态为 $(1,27,1)$ 和 3 份 $(3,1,1)$。由于共有 27 个不动点 $\left(\det{}'(1-g) = 27\right)$，$g$ 扭曲扇区总共给出 27 份 $(1,27,1)$ 和 81 份 $(3,1,1)$。同理，在 g^2 扭曲扇区我们得到 27 份 $(1,27,1)$ 和 81 份 $\left(\bar{3},1,1\right)$。

To conclude, in this chapter we have discussed toroidal as well as symmetric orbifold compactifications where the orbifold group acts symmetrically on the left and right movers. This has been extended to asymmetric orbifold constructions where the orbifold group acts asymmetrically on the left and right movers (Ref. [5]) which allows many more possibilities. However this has not been discussed in this chapter. Similarly we have not discussed the computations of string effective action terms that involves computing correlation functions of, in particular, twisted states. This has been discussed in detail in Ref. [6].

总而言之，本章我们讨论了环面紧致化与对称轨形紧致化，这类紧致化中轨形群对称作用于左行模和右行模。该构造已被推广到不对称轨形构造: 轨形群不对称作用于左行模与右行模 (参考文献 [5])，这带来了更多可能性，但本章不对其展开讨论。同理，我们也未讨论弦有效作用量项的计算，这类计算尤其涉及扭曲态的关联函数计算，相关内容已在参考文献 [6] 中有详细讨论。

Appendix A

附录 A

We consider the following lattice sum

我们考虑如下格点和

$$Z_I\left(v,w,\tau,\bar{\tau}\right) = \sum_{P\in I} q^{\frac{1}{2}\left(P_L+v_L\right)^2}\bar{q}^{\frac{1}{2}\left(P_R+v_R\right)^2}e^{2\pi i(P+v).w} \tag{30}$$

where $P = (P_L,P_R)$, $v = (v_L,v_R)$, $w = (w_L,w_R)$ and the dot products are with Lorentzian metric $P.v = P_L.v_L - P_R.v_R$ etc. I is an even integral lattice with Lorentzian metric. Now we do a Poisson resummation. Let

其中 $P = (P_L,P_R)$, $v = (v_L,v_R)$, $w = (w_L,w_R)$ 与内积都采用洛伦兹度规 $P.v = P_L.v_L - P_R.v_R$ 等。I 是配备洛伦兹度规的偶整格。现在我们做泊松重求和。令

$$f(x) = \sum_{P\in I} q^{\frac{1}{2}\left(P_L+v_L+x_L\right)^2}\bar{q}^{\frac{1}{2}\left(P_R+v_R+x_R\right)^2}e^{2\pi i(P+v+x).w} \tag{31}$$

where x is an arbitrary vector in a basic cell B with same dimension as that of I (let us denote by d the dimension of I). Then $f(x+P') = f(x)$ for all $P' \in I$. This means that we can make a Fourier expansion

其中 x 是基胞 B 中的任意向量，基胞与 I 维度相同 (我们用 d 表示 I 的维度)。则对所有 $P' \in I$ 满足 $f(x+P') = f(x)$ 。这说明我们可以做傅里叶展开

$$f(x) = \sum_{k \in I^*} f_k e^{-2\pi i k.x} \tag{32}$$

where I^* is dual of I with Lorentzian signature and

其中 I^* 是具有洛伦兹符号差的 I 的对偶格，且

$$f_k = \frac{1}{Vol} \int dx f(x) e^{2\pi i k.x} \tag{33}$$

where $Vol = \sqrt{\left|\frac{I^*}{I}\right|}$ is the volume of a basic cell B . Note that since I is integral, it is contained in I^* and $\left|\frac{I^*}{I}\right|$ is just the number of elements in $\frac{I^*}{I}$. This integral can be carried out as follows

其中 $Vol = \sqrt{\left|\frac{I^*}{I}\right|}$ 是基胞 B 的体积。注意由于 I 是整格，它包含于 I^* 中，且 $\left|\frac{I^*}{I}\right|$ 就是 $\frac{I^*}{I}$ 中的元素个数。该积分可按下文计算

$$\begin{aligned}
f_k &= \frac{1}{Vol} \int_B dx \sum_{P \in I} e^{i\pi\tau(P_L+v_L+x_L)^2} e^{-i\pi\bar{\tau}(P_R+v_R+x_R)^2} e^{2\pi i(P+v+x)\cdot w} e^{2\pi i k\cdot x} \\
&= \frac{1}{Vol} e^{-2\pi i k\cdot v} \int_B dx \sum_{P \in I} e^{i\pi\tau(P_L+v_L+x_L)^2} e^{-i\pi\bar{\tau}(P_R+v_R+x_R)^2} e^{2\pi i(P+v+x)\cdot w} e^{2\pi i k\cdot(P+v+x)} \\
&= \frac{1}{Vol} e^{-2\pi i k.v} \int_{R^d} dx e^{i\pi\tau x_L^2} e^{-i\pi\bar{\tau} x_R^2} e^{2\pi i x.w} e^{2\pi i k.x} \\
&= \frac{1}{Vol} e^{2\pi i k.v} \int_{R^d} dx e^{i\pi\tau\left(x_L+\frac{(w_L+k_L)}{\tau}\right)^2} e^{-i\pi\bar{\tau}\left(x_R+\frac{(w_R+k_R)}{\bar{\tau}}\right)^2} e^{-i\pi\frac{(w_L+k_L)^2}{\tau}} e^{-i\pi\frac{(w_R+k_R)^2}{\bar{\tau}}} \\
&= \frac{1}{Vol} \frac{1}{(-i\tau)^{d_L/2}(i\bar{\tau})^{d_R/2}} e^{2\pi i k\cdot v} e^{i\pi\tau'(w_L+k_L)^2} e^{-i\pi\bar{\tau}'(w_R+k_R)^2}
\end{aligned} \tag{34}$$

where $\tau' = -1/\tau$. Substituting this back in Eq. (32) and setting $x = 0$ we find

其中 $\tau' = -1/\tau$ 。将其代回式 (32) 并令 $x = 0$ ，我们得到

$$\begin{aligned}
Z_I(v,w)(\tau,\bar{\tau}) &= \sum_{k \in I^*} \frac{1}{Vol} \frac{1}{(-i\tau)^{d_L/2}(i\bar{\tau})^{d_R/2}} e^{2\pi i k\cdot v} e^{i\pi\tau'(w_L+k_L)^2} e^{-i\pi\bar{\tau}'(w_R+k_R)^2} \\
&= \frac{1}{Vol} \frac{1}{(-i\tau)^{d_L/2}(i\bar{\tau})^{d_R/2}} \sum_{v' \in I^*/I} \\
&\quad \sum_{P \in I} q'^{\frac{1}{2}(P_L+v'_L+w_L)^2} \bar{q}'^{\frac{1}{2}(P_R+v'_R+w_R)^2} e^{2\pi i v\cdot v'}
\end{aligned} \tag{35}$$

where $q' = e^{i\pi\tau'}$ and in the last line the sum over $k \in I^*$ is expressed in terms of sum over $v' \in I^*/I$ and $P \in I$ by writing $k = v' + P$.

其中 $q' = e^{i\pi\tau'}$ ，最后一行通过改写 $k = v' + P$ ，将对 $k \in I^*$ 的求和转化为对 $v' \in I^*/I$ 和 $P \in I$ 的求和。

Note that since I is integral, it is contained in I^* and $\left|\frac{I^*}{I}\right|$ just the number of elements in $\frac{I^*}{I}$.

注意由于 I 是整格，它包含于 I^* 中，且 $\left|\frac{I^*}{I}\right|$ 就是 $\frac{I^*}{I}$ 中的元素个数。

Including also the partition function of the oscillator modes of d_L left moving scalars and d_R right moving scalars, which give contribution $1/\eta^{d_L}\overline{\eta}^{d_R}$ and using the fact that $\eta(\tau) = \sqrt{-i\tau'}\eta(\tau')$ we see that all the explicit τ and $\overline{\tau}$ factors cancel

计入 d_L 个左动标量与 d_R 个右动标量的振子模配分函数 (其贡献为 $1/\eta^{d_L}\overline{\eta}^{d_R}$)，并利用性质 $\eta(\tau) = \sqrt{-i\tau'}\eta(\tau')$ ，我们可以看到所有显式的 τ 与 $\overline{\tau}$ 因子都抵消了

$$\frac{Z_I(v,w)\left(\tau,\overline{\tau}\right)}{\eta(\tau)^{d_L}\eta\left(\overline{\tau}\right)^{d_R}} = \frac{1}{Vol}\sum_{v'\in I^*/I}\frac{Z_I(w+v',-v)\left(\tau',\overline{\tau}'\right)}{\eta(\tau')^{d_L}\eta\left(\overline{\tau}'\right)^{d_R}} \tag{36}$$

Case when I is self dual

I 为自对偶的情形

The above discussion was for I which is even and integral. In the special case when I is self dual lattice, i.e., $I^* = I$, then $\mathrm{Vol} = \sqrt{\left|\frac{I^*}{I}\right|} = 1$ and the sum over v' is trivial, so that

上述讨论针对偶整 I 。在 I 为自对偶格的特殊情形下，即满足 $I^* = I$ ，此时有 $\mathrm{Vol} = \sqrt{\left|\frac{I^*}{I}\right|} = 1$ ，且对 v' 的求和是平凡的，因此

$$\begin{aligned} Z_I(v,w)\left(\tau,\overline{\tau}\right) &= \frac{1}{(-i\tau)^{d_L/2}\left(i\overline{\tau}\right)^{d_R/2}}\sum_{P\in I} q'^{\frac{1}{2}(P_+w_L)^2}\bar{q}'^{\frac{1}{2}(P_R+w_R)^2}e^{-2\pi i v\cdot P} \\ &= \frac{1}{(-i\tau)^{d_L/2}\left(i\overline{\tau}\right)^{d_R/2}}Z_I(w,-v)\left(\tau',\overline{\tau'}\right) \end{aligned} \tag{37}$$

Including also the oscillator part we find that when I is self dual

纳入振子部分后，我们得到当 I 为自对偶时

$$\frac{Z_I(v,w)\left(\tau,\overline{\tau}\right)}{\eta(\tau)^{d_L}\eta\left(\overline{\tau}\right)^{d_R}} = \frac{Z_I(w,-v)\left(\tau',\overline{\tau}'\right)}{\eta(\tau')^{d_L}\eta\left(\overline{\tau}'\right)^{d_R}} \tag{38}$$

Appendix B

附录 B

Here we will review definitions and some properties of θ -functions. The sum form of the θ functions are:

此处我们将回顾 θ 函数的定义与部分性质。θ 函数的和形式如下:

$$\theta_{(\alpha,\beta)}(z,\tau) = \sum_n e^{i\pi\tau(n+\alpha)^2 + 2i\pi(n+\alpha)(z+\beta)} \tag{39}$$

They can also be written in the product form. For $\alpha = 0$, $\vartheta_{(0,\beta)}(z,\tau)$ is

它们也可以写成乘积形式。对于 $\alpha = 0, \vartheta_{(0,\beta)}(z,\tau)$，其乘积形式为

$$\prod_{n>0}(1-q^n)\left(1+e^{2\pi i(z+\beta)}q^{n-\frac{1}{2}}\right)\left(1+e^{-2\pi i(z+\beta)}q^{n-\frac{1}{2}}\right) \tag{40}$$

and for $\alpha = 1/2$ the product form for $\vartheta_{\left(\frac{1}{2},\beta\right)}(z,\tau)$ is

对于 $\alpha = 1/2$，$\vartheta_{\left(\frac{1}{2},\beta\right)}(z,\tau)$ 的乘积形式为

$$2\cos(\pi(z+\beta))\prod_{n>0}(1-q^n)\left(1+e^{2\pi i(z+\beta)}q^n\right)\left(1+e^{-2\pi i(z+\beta)}q^n\right) \tag{41}$$

Under $z \to z + m_1\tau + m_2$ for $m_1, m_2 \in \mathcal{Z}$, they transform as

在 $z \to z + m_1\tau + m_2$ 作用于 $m_1, m_2 \in \mathcal{Z}$ 时，它们的变换为

$$\vartheta_{(\alpha,\beta)}(z + m_1\tau + m_2, \tau) = e^{-\pi i m_1^2\tau - 2\pi i m_1 z + 2\pi i(m_2\alpha - m_1\beta)}\vartheta_{(\alpha,\beta)}(z,\tau) \tag{42}$$

Under T and S transformations:

在 T 和 S 变换下:

(1) For $\tau' = \tau + 1$:

(1) 对于 $\tau' = \tau + 1$:

$$\vartheta_{(\alpha,\beta)}(z,\tau) = e^{\pi i\alpha(\alpha-1)}\vartheta_{\left(\alpha,\beta-\alpha+\frac{1}{2}\right)}(z,\tau') \tag{43}$$

(2) For $\tau' = -1/\tau$:

(2) 对于 $\tau' = -1/\tau$:

$$\vartheta_{(\alpha,\beta)}(z,\tau) = \sqrt{-i\tau'}e^{\pi i z^2 \tau'} e^{2\pi i\alpha\beta} \vartheta_{(\beta,-\alpha)}(z\tau',\tau') \tag{44}$$

Finally the Dedekind η function transforms as :

最后，戴德金 η 函数的变换为:

For $\tau = \tau' + 1$

对于 $\tau = \tau' + 1$

$$\eta(\tau) = e^{-i\pi/12}\eta(\tau') \tag{45}$$

For $\tau = -1/\tau'$

对于 $\tau = -1/\tau'$

$$\eta(\tau) = \sqrt{-i\tau'}\eta(\tau') \tag{46}$$

Appendix C

附录 C

Here we will discuss the bosonization of world sheet fermions. Consider two real left moving fermions say ψ_1 and ψ_2 . Virasoro central charge of this system is $c = 1$. We can combine them in terms of a complex fermion $\psi = \psi_1 + i\psi_2$ and $\overline{\psi} = \psi_1 - i\psi_2$. The operator $\overline{\psi}\psi$ has dimension one and in fact is a $U(1)$ current say J , It is easy to see using OPE between J and ψ and $\overline{\psi}$ they carry +1 and -1 $U(1)$ charges, respectively.

我们在此讨论世界面费米子的玻色化。考虑两个左行实费米子，即 ψ_1 和 ψ_2 。该系统的维拉索罗中心荷为 $c = 1$ 。我们可以将它们组合为复费米子 $\psi = \psi_1 + i\psi_2$ 和 $\overline{\psi} = \psi_1 - i\psi_2$ 。算符 $\overline{\psi}\psi$ 的维度为 1，实际上它是一个 $U(1)$ 流，即 J 。利用 J 与 ψ 、$\overline{\psi}$ 之间的算符乘积展开容易看出，它们分别携带 +1 和-1 的 $U(1)$ 电荷。

We can express all the above in terms of a world sheet scalar ϕ with the identification $\partial\phi$ with $\overline{\psi}\psi$, and vertex operators $e^{i\phi}$ and $e^{-i\phi}$ with ψ and $\overline{\psi}$, respectively.

我们可以利用世界面标量 ϕ 将上述所有内容重新表述: 将 $\partial\phi$ 等同于 $\overline{\psi}\psi$ ，顶点算符 $e^{i\phi}$ 和 $e^{-i\phi}$ 分别等同于 ψ 和 $\overline{\psi}$ 。

We can now identify the states in the fermion formulation with that in bosonic formulation as follows. In the NS sector the fermion partition function is $q^{-\frac{1}{24}} \prod_{n>0} \left(1 + q^{n-\frac{1}{2}}\right)^2$ and in the GSO projected sector $q^{-\frac{1}{24}} \prod_{n>0} \left(1 - q^{n-\frac{1}{2}}\right)^2$, where we have included the zero point shift. By using the identities given in Appendix

B in (39),(40) and (41) is the same as $\frac{\sum_n q^{\frac{1}{2}n^2}}{\eta}$ and $\frac{\sum_n (-1)^n q^{\frac{1}{2}n^2}}{\eta}$. The first is exactly the partition function of a single boson with momenta being all integers n and the second where odd n terms come with minus sign. Adding the two terms or taking the difference between the two, in the bosonic language it amounts to summing over even momenta or odd momenta. Similarly in the Ramond sector using the relations in Appendix B, one finds that in the bosonic language it corresponds to summing over momenta that are 1/2 mod integers.

现在我们可以将费米子表述中的态对应到玻色子表述的态，如下所述。在 NS 区，费米子配分函数为 $q^{-\frac{1}{24}}\prod_{n>0}\left(1+q^{n-\frac{1}{2}}\right)^2$，GSO 投影后的区为 $q^{-\frac{1}{24}}\prod_{n>0}\left(1-q^{n-\frac{1}{2}}\right)^2$，此处我们已经计入了零点移位。利用附录 B 中 (39)、(40) 和 (41) 给出的恒等式，上述配分函数等同于 $\frac{\sum_n q^{\frac{1}{2}n^2}}{\eta}$ 和 $\frac{\sum_n (-1)^n q^{\frac{1}{2}n^2}}{\eta}$。第一个正是单个玻色子的配分函数，其动量均为整数 n，第二个中奇数 n 项带有负号。将两项相加，或是取两项的差，在玻色子语言中对应对偶动量或奇动量求和。类似地，在拉蒙德区利用附录 B 中的关系，我们可知在玻色子语言中，它对应对满足 1/2 模整数条件的动量求和。

Now let us look at the effect of a rotation by an angle η : $(\psi_1,\psi_2)\to(\cos\eta\psi_1+\ \sin\eta\psi_2,-\sin\eta\psi_1+\cos\eta\psi_2)$. This is the same as η : $\left(\psi,\overline{\psi}\right)\to\left(e^{i\eta}\psi,e^{-i\eta}\overline{\psi}\right)$. In terms of boson ϕ, the effect of this rotation by η is $e^{i\phi}\to e^{i\eta}e^{i\phi}$ which amounts to a shift $\phi\to\phi+\eta$. Thus rotation in the language of fermions becomes a shift in the bosonized version.

现在我们来看转角为 η：$(\psi_1,\psi_2)\to(\cos\eta\psi_1+\ \sin\eta\psi_2,-\sin\eta\psi_1+\cos\eta\psi_2)$ 的旋转的效应。这等价于 η：$\left(\psi,\overline{\psi}\right)\to\left(e^{i\eta}\psi,e^{-i\eta}\overline{\psi}\right)$。对于玻色子 ϕ，转角为 η 的旋转的效应是 $e^{i\phi}\to e^{i\eta}e^{i\phi}$，这对应一个平移 $\phi\to\phi+\eta$。因此费米子语言中的旋转，在玻色化表述中成为平移。

In the orbifold context we had rotation in the three internal planes by $\frac{2\pi}{N}(0,r_1,r_2,r_3)$ where the first entry refers to the transverse light cone plane in 4-dimension. On each plane their right moving fermion partners will also be rotated by the same amount to preserve world sheet supersymmetry. When we bosonize the pairs of fermions in each plane, this rotation acts as shifts $(\phi_1,\phi_2,\phi_3,\phi_4)\to(\phi_1,\phi_2,\phi_3,\phi_4)+\frac{2\pi}{N}(0,r_1,r_2,r_3)$.

在轨形背景下，我们有三个内平面的转动，转动量为 $\frac{2\pi}{N}(0,r_1,r_2,r_3)$，其中第一个条目对应四维中的横向光锥平面。为了保持世界面超对称，每个平面上的右行费米子伙伴也会被转动相同角度。当我们对每个平面中的费米子对玻色化时，该转动表现为平移 $(\phi_1,\phi_2,\phi_3,\phi_4)\to(\phi_1,\phi_2,\phi_3,\phi_4)+\frac{2\pi}{N}(0,r_1,r_2,r_3)$.

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